STANFORD RESEARCH INSTITUTE
MENLO PARK, CALIFORNIA

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August 1964

Technical Report 205

TEMPERATURE EFFECTS ON THE HIGH-FREQUENCY STABILITY OF A CYLINDRICAL, RELATIVISTIC ELECTRON BEAM PENETRATING A PLASMA: II

Prepared for:

OFFICE OF NAVAL RESEARCH
WASHINGTON, D.C.

CONTRACT Nonr-2728(00)

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ABSTRACT

This report continues the study of the effect of temperature upon the propagation of high frequency, small amplitude disturbances in a relativistic beam-plasma system of finite radius.

The current response of the beam is calculated in this report by using the accurate "betatron oscillation" orbits of the beam particles. The effect of the beam's self-magnetic field on the plasma is neglected.

Maxwell's equations are investigated for an arbitrary normal mode. Both beam and plasma temperatures are included in deriving the dispersion relations for the $\ell = 0$ normal mode. For $\ell \neq 0$ Maxwell's equations are easily uncoupled only for a cold plasma. Dispersion relations are given for this case for an infinite plasma only.

When the plasma temperature is not zero, uncoupling of Maxwell's equations for $\ell \neq 0$ leads to a sixth order differential equation for $E_z(r)$. For this case some special modes are investigated. In particular, it is shown that pure transverse waves are possible only for $\ell = 0$. However, pure longitudinal waves are found to be possible for all ℓ, provided the relevant dispersion relations have solutions.

CONTENTS

I INTRODUCTION

This report continues the study of temperature effects upon the high frequency stability of a beam-plasma system. (The first report in this series[1*] will henceforth be referred to as TI).

In Section II, the zero-order orbits of the beam particles are derived taking into account the self-magnetic field of the beam.

In Section III, the current response of the beam is calculated for an arbitrary normal mode using the orbits derived in Section II. In particular, it is shown that the high frequency criterion, $|\omega - kV_0|^2 \gg \omega_\beta^2$, reproduces the results derived in TI—where straight-line motion was assumed for all particles. Also in Section III the plasma current is calculated for an arbitrary normal mode. The effect on the plasma of the beam's magnetic field is neglected. This is shown to correspond to the following additional restriction upon the frequency:

$$\frac{|\omega + i\nu_e|}{\omega_\beta} \gg \frac{\gamma_0 \omega_\beta r_0}{V_0} \quad .$$

Maxwell's equations for an arbitrary normal mode are investigated in Section IV. For the $\ell = 0$ normal mode, the equations are readily uncoupled as in TI. However for $\ell \neq 0$, the equations are easily solvable only for a cold plasma. In general, for non-zero plasma temperature, uncoupling of the equations results in a sixth order differential equation for $E_z(r)$.

In Section V, dispersion relations including both beam and plasma temperatures are derived for the $\ell = 0$ normal mode. Dispersion relations are given for an infinite plasma, a plasma bounded by a conducting wall, and a finite plasma beyond which is vacuum.

Dispersion relations for a cold, infinite plasma are derived for arbitrary ℓ in Section VI. In Section VII, some special modes are investigated for non-zero plasma temperature and $\ell \neq 0$. The pure transverse

* References are listed at the end of the report.

and pure longitudinal modes are investigated. It is found that pure transverse waves are only possible for $\ell = 0$. However, for arbitrary ℓ (including $\ell = 0$) pure longitudinal waves are possible, provided the relevant dispersion relations have solutions. In a preliminary analysis of these relations, unstable solutions are found only for $\ell = 0$ and are equivalent to the case of an infinite beam in an infinite, zero-temperature plasma.

II THE ZERO-ORDER ORBITS

For an electron beam of equilibrium density $n_{01}(r)$ moving with a mean drift velocity $\vec{V}_0 = V_0\hat{z}$, the macroscopic current is

$$\vec{J}^B_0(r) = -en_{01}(r)\vec{V}_0 \tag{1}$$

where $r = [x^2 + y^2]^{1/2}$. The magnetic field due to this current is given by

$$\nabla \times \vec{B}_0 = \frac{4\pi}{c}\vec{J}_0 \quad . \tag{2a}$$

That is,

$$\frac{1}{r}\frac{d}{dr}(rB_{0\theta}) = -\frac{4\pi en_{01}(r)V_0}{c} \quad . \tag{2b}$$

Therefore

$$\vec{B}_0(r) = -\frac{4\pi eV_0}{c}\frac{1}{r}\int_0^r n_{01}(r')r'dr'\hat{\theta} \quad . \tag{3}$$

Examples:

(1) The square beam case—$n_{01}(r) = n_{01}$ for $r < r_0$ and $n_{01}(r) = 0$ for $r > r_0$. Letting $b = 2\pi en_{01}V_0/c$ then

$$r < r_0: \quad \vec{B}_0 = -br\hat{\theta} = b(y\hat{x} - x\hat{y}) \tag{4a}$$

$$r > r_0: \quad \vec{B}_0 = -b\frac{r_0^2}{r}\hat{\theta} = b\left(\frac{r_0}{r}\right)^2(y\hat{x} - x\hat{y}) \quad . \tag{4b}$$

(2) $n_{01}(r) = n_{01}$ for $r < r_0$ and $n_{01}(r) = n_{01}r_0/r$ for $r > r_0$:

$$r < r_0: \quad \vec{B}_0 = -br\hat{\theta} = b(y\hat{x} - x\hat{y}) \tag{4c}$$

$$r > r_0: \quad \vec{B}_0 = -b\frac{r_0^2}{r}\left[2\frac{r}{r_0} - 1\right]\hat{\theta} = b\left(\frac{r_0}{r}\right)^2\left[2\frac{r}{r_0} - 1\right](y\hat{x} - x\hat{y}) \quad . \tag{4d}$$

(3) Gaussian—$n_{01}(r) \; = \; n_{01} \exp(-r^2/r_0^2)$.

$$\vec{B}_0 = b\left(\frac{r_0}{r}\right)^2\left[1 - e^{-r^2/r_0^2}\right](y\hat{x} - x\hat{y}) \quad . \tag{4e}$$

To take advantage of its obvious simplicity only the square beam, Example (1) will be considered here.

The beam particle orbits are determined by Newton's second law using the two-mass approximation:

$$\gamma_0^3 m \frac{d\vec{V}'_z}{dt'} + \gamma_0 m \frac{d\vec{v}'_\perp}{dt'} \simeq -\frac{e}{c}\vec{V}' \times \vec{B}_0 \quad . \tag{5}$$

At time $t' = t$, $\vec{V}' = (V_0 + v_z)\hat{z} + \vec{v}_\perp$ and $\vec{x}' = z\hat{z} + \vec{r}$. . Using Eq. (4a), Eq. (5) becomes

$$\frac{dV'_z}{dt'} = \frac{\omega_\beta^2}{\gamma_0^2 V_0}(x'v'_x + y'v'_y) \tag{6a}$$

$$\frac{d\vec{v}'_\perp}{dt'} = -\frac{\omega_\beta^2}{V_0}V'_z\vec{r}' \tag{6b}$$

where

$$\omega_\beta^2 = \frac{2\pi e^2 n_{01} V_0^2}{\gamma_0 m c^2} \quad .$$

The constants of the motion, which are easily found from Eqs. (6a,b), are

$$T = \gamma_0^2 V'^2_z + v'^2_\perp \tag{7a}$$

and

$$L_z = x'v'_y - y'v'_x = r'v'_\theta \quad . \tag{7b}$$

Also, since the beam particles are constrained to be within r_0, the boundary condition, $v'_r = 0$ at $r' = r_0$, is imposed. Noting that Eq. (6a) can be written as

$$\frac{dV'_z}{dt'} = \frac{\omega_\beta^2}{2\gamma_0^2 V_0} \frac{dr'^2}{dt'} \tag{8}$$

then, using Eqs. (7a) and (7b) and the condition $v'_r(r_0) = 0$,

$$V'_z = \frac{\omega_\beta^2}{2\gamma_0^2 V_0}(r'^2 - r_0^2) + V_0 G^2 \tag{9}$$

where

$$G^2 = \frac{1}{\gamma_0 V_0}\left[T - \frac{1}{r_0^2} L_z^2\right]^{1/2} .$$

An equation for v'_r in terms of r' is obtained by using Eqs. (7a,b) and (9):

$$(v'_r)^2 = (r'^2 - r_0^2)\left[-\frac{\omega_\beta^4}{4\gamma_0^2 V_0^2}(r'^2 - r_0^2) - \omega_\beta^2 G^2 + \frac{1}{(r' r_0)^2} L_z^2\right] . \tag{10}$$

Since $(v'_r)^2 = (dr'/dt')^2$ then, letting $\tau = t - t'$ so that $r' = r$ when $\tau = 0$,

$$\int_0^\tau d\tau = \int_r^{r'} \frac{u\,du}{r\left[r_0^2 - U^2\right]^{1/2}\left[-\frac{\omega_\beta^4}{4\gamma_0^2 V_0^2} U^4 + \omega_\beta^2 G^2\left(\frac{\omega_\beta^2 r_0^2}{4\gamma_0^2 V_0^2 G^2} - 1\right)U^2 + \frac{1}{r_0^2} L_z^2\right]^{1/2}} . \tag{11}$$

Defining the dimensionless quantities

$$q^2 = \frac{1}{4}\frac{\omega_\beta^2 r_0^2}{\gamma_0^2 V_0^2 G^2} \tag{12}$$

and

$$L^2 = \frac{L_z^2}{\omega_\beta^2 V_0^4 G^2} \quad , \tag{13}$$

and changing the variable of integration to $w = u^2$, Eq. (11) can be put into the form

$$\omega_\beta G\tau = \frac{1}{2}\frac{r_0}{q}\int_{r^2}^{r'^2} \frac{dw}{[r_0^2 - w]^{1/2}[w - D_+]^{1/2}[w - D_-]^{1/2}} \tag{14}$$

where

$$D_\pm = \frac{1}{2}\frac{r_0^2}{q^2}(q^2 - 1) \pm \frac{r_0^2}{q^2}[(1 - q^2)^2 + 4q^2L^2]^{1/2} \quad . \tag{15}$$

It will now be assumed that $q^2 < 1$. Thus

$$D_+ = r_0^2 L^2 + 0(q^2) \tag{15a}$$

$$D_- = r_0^2\left(1 - \frac{1}{q^2}\right) - r_0^2 L^2 + 0(q^2) \quad . \tag{15b}$$

The first case to be treated will be that of zero angular momentum; *i.e.*, $L^2 = 0$. Then $D_+ = 0$ and $D_- = r_0^2(1 - 1/q^2) < 0$ so the integral in Eq. (14) can be done using No. 544 of Pierce[2]. Hence

$$\omega_\beta G\tau = \frac{r_0}{q}[r_0^2 - D_-]^{-1/2}\{sn^{-1}(\Psi,\Lambda) - sn^{-1}(\Psi',\Lambda)\} \tag{16}$$

where

$$\Psi^2 = \frac{r_0^2 - r^2}{r_0^2 - D_+} = \frac{r_0^2 - r^2}{r_0^2}$$

$$\Psi'^2 = \frac{r_0^2 - r'^2}{r_0^2 - D_+} = \frac{r_0^2 - r'^2}{r_0^2}$$

$$\Lambda^2 = \frac{r_0^2 - D_+}{r_0^2 - D_-} = q^2$$

and

$$\frac{r_0}{q}\,[r_0^2 - D_-]^{-\frac{1}{2}} = 1 \quad .$$

Since $0 \leq \Psi \leq 1$ then $sn^{-1}(\Psi,\Lambda) = F(\sin^{-1}\Psi,\Lambda)$ where $F(\phi,\Lambda)$ is the elliptic integral of the first kind. Therefore Eq. (16) becomes

$$\omega_\beta G\tau = F(\sin^{-1}\Psi,\Lambda) - F(\sin^{-1}\Psi',\Lambda) \quad . \tag{17}$$

Now for $\Lambda^2 < 1$,

$$F(\phi,\Lambda) = \phi + [\phi - \sin\phi\cos\phi]\,\frac{1}{4}\Lambda^2 + \ldots \quad ,$$

so, including terms up to first order in q^2, Eq. (17) becomes

$$\omega_\beta G\tau = \sin^{-1}\Psi - \sin^{-1}\Psi' + \frac{1}{4}q^2\{\sin^{-1}\Psi - \sin^{-1}\Psi' - \Psi[1 - \Psi^2]^{\frac{1}{2}} + \Psi'[1 - \Psi'^2]^{\frac{1}{2}}\} + \ldots \quad . \tag{18}$$

Consider Eq. (18) to zero order in q^2. It can be written as

$$\omega_\beta G\tau = \sin^{-1}\left[\left(\frac{r_0^2 - r^2}{r_0^2}\right)^{\frac{1}{2}}\right] - \sin^{-1}\left[\left(\frac{r_0^2 - r'^2}{r_0^2}\right)^{\frac{1}{2}}\right] \quad . \tag{19}$$

Using the identities

$$\sin^{-1}x = \frac{1}{2}\pi - \cos^{-1}x$$

and

$$\cos^{-1}y - \cos^{-1}x = \cos^{-1}\left\{xy + [(1 - x^2)(1 - y^2)]^{\frac{1}{2}}\right\}$$

then Eq. (19) becomes

$$\left(\frac{r'}{r_0}\right)^2 - 2\,\frac{rr'}{r_0^2}\cos\omega_\beta G\tau + \frac{r^2}{r_0^2} - \sin^2\omega_\beta G\tau = 0 \tag{20a}$$

or, finally,

$$r'^2 = [r\cos\omega_\beta G\tau \pm (r_0^2 - r^2)^{\frac{1}{2}}\sin\omega_\beta G\tau]^2 \quad . \tag{20b}$$

Equation (20b) leads to simple orbit equations when expressed in terms of the Cartesian coordinates, x' and y'. Since $L^2 = 0$ implies $V'_\theta = 0$, then $\theta' = \theta$. So $x' = r' \cos\theta$ and $y' = r' \sin\theta$, where

$$X' = r \cos\theta \cos \omega_\beta G\tau \pm (r_0^2 - r^2)^{1/2} \cos\theta \sin \omega_\beta G\tau \qquad (21a)$$

$$Y' = r \sin\theta \cos \omega_\beta G\tau \pm (r_0^2 - r^2)^{1/2} \sin\theta \sin \omega_\beta G\tau \quad . \qquad (21b)$$

From Eqs. (21a,b)

$$v'_x = \omega_\beta G r \cos\theta \sin \omega_\beta G\tau \mp \omega_\beta G (r_0^2 - r^2)^{1/2} \cos\theta \cos \omega_\beta G\tau \qquad (22a)$$

$$v'_y = \omega_\beta G r \sin\theta \sin \omega_\beta G\tau \mp \omega_\beta G (r_0^2 - r^2)^{1/2} \sin\theta \cos \omega_\beta G\tau \quad . \qquad (22b)$$

Setting $\tau = 0$ in Eq. (22) leads to

$$v_x^2 = \omega_\beta^2 G^2 (r_0^2 - r^2) \cos^2\theta \qquad (23a)$$

$$v_y^2 = \omega_\beta^2 G^2 (r_0^2 - r^2) \sin^2\theta \qquad (23b)$$

or, equivalently,

$$v_\perp^2 \equiv v_x^2 + v_y^2 = \omega_\beta^2 G^2 (r_0^2 - r^2) \quad . \qquad (23c)$$

Also, since $L^2 = 0$,

$$G^2 = \left[\frac{1}{\gamma_0^2 V_0^2} T\right]^{1/2} = \left[\frac{V_z^2}{V_0^2} + \frac{v_\perp^2}{\gamma_0^2 V_0^2}\right]^{1/2} \quad . \qquad (24a)$$

Using Eq. (23c)

$$\frac{v_\perp^2}{\gamma_0^2 V_0^2} = \frac{\omega_\beta^2 (r_0^2 - r^2)}{\gamma_0^2 V_0^2} G^2$$

which is of the order of q^2 since G^2 is of the order of unity. So to zero order in q^2,

$$G^2 \simeq \frac{V_z}{V_0} \quad . \qquad (24b)$$

Also note that, using Eq. (24b), the first term in Eq. (9) is of the order of q^2 relative to the second term.

In summary, the zero-order orbits for the beam particles for zero angular momentum and to zero order in $q^2 = 1/4[(\omega_\beta^2 V_0^2)/(\gamma_0^2 V_0^2 G^2)]$ (it is actually quite evident from D_+ and D_- that these results are also true for non-zero angular momentum as long as the angular momentum is small enough such that L^2 is of the same order of magnitude as q^2), are

$$V_z' = V_0 + v_z$$

$$v_x' = \omega_0 x \sin \omega_0 \tau + v_x \cos \omega_0 \tau$$

$$v_y' = \omega_0 y \sin \omega_0 \tau + v_y \cos \omega_0 \tau \tag{25}$$

and

$$z' = z - (V_0 + v_z)\tau$$

$$x' = x \cos \omega_0 \tau - \frac{1}{\omega_0} v_x \sin \omega_0 \tau$$

$$y' = y \cos \omega_0 \tau - \frac{1}{\omega_0} v_y \sin \omega_0 \tau \tag{26}$$

where

$$\omega_0 = \omega_\beta G \quad \text{and} \quad G^2 = 1 + \frac{v_z}{V_0} \quad .$$

From Eqs. (25) and (26)

$$v_r'^2 + \omega_0^2 r'^2 = v_r^2 + \omega_0^2 r^2 = \omega_0^2 r_0^2$$

when $v_\theta' \approx 0$. Therefore

$$v_r'^2 + \omega_0^2(r'^2 - r_0^2) \ , \quad v_z' \ , \quad \text{and} \quad v_\theta'$$

can be taken as the constants of the motion described by Eqs. (25) and (26).

It is of interest to inquire whether Eqs. (25) and (26) correctly describe the beam particle orbits for large angular momentum; *i.e.*,

$L^2 >> q^2$. For the case of $4q^2L^2 >> 1$, $D_\pm \approx \pm r_0^2 L/q$. By the same general method as above, the orbit equation, replacing Eq. (20b), is now found to be

$$r'^2 \approx \frac{L}{2q} r_0^2 + \sqrt{2q}\left[\sqrt{\frac{L}{2q}}\, r_0 \cos \omega_\beta G_L \tau \pm (r_0^2 - r^2)^{1/2} \sin \omega_\beta G_L \tau\right]^2 \qquad (27)$$

where

$$G_L = \sqrt{2qL}\, G \quad .$$

It is obvious that Eqs. (25) and (26) do not satisfy Eq. (27) and are, therefore, the beam particle orbit equations only for small enough angular momentum such that $L^2 \lesssim q^2$.

It is quite obvious that Eqs. (25) and (26) are readily obtained from Eqs. (6a,b) by setting the right-hand side of Eq. (6a) equal to zero (which is equivalent to letting $\gamma_0^2 \to \infty$). There are two advantages to the rather elaborate derivation presented here:

(1) it has been shown that the accuracy of Eqs. (25) and (26) depends upon the smallness of the parameter, q^2, and

(2) Eqs. (25) and (26) are only valid for small enough angular momentum such that $L^2 \lesssim q^2 << 1$, or equivalently, $v_\theta^2/\gamma_0^2 V_0^2 << 4q^2 << 1$.

Point (1) made above has some relevance to the work of Mjolsness[3] in which Eqs. (25) and (26) are used as the beam particle orbits. Later in his work he uses $2\gamma_0 q$ as an expansion parameter to solve his integral equations. Therefore, care must be taken in considering some of the cases discussed by Mjolsness, such as those in which $2\gamma_0 q$ is assumed larger than certain other terms, since this may be inconsistent with the use of Eqs. (25) and (26). However, it should be noted that Mjolsness derives the macroscopic "hose instability" from the microscopic point of view (*i.e.*, by using the Boltzmann equation to describe the beam particles) by setting $q = 0$. This case is obviously completely consistent with the use of Eqs. (25) and (26) as the equations for the zero-order orbits.

The use of Eqs. (25) and (26) introduces sufficient complexity into the calculation of the beam current so that the generalization to arbitrary L^2 or the inclusion of higher order terms in q^2 is not warranted at this time.

The restriction to zero in q^2 requires that $q^2 << 1$ or, since G^2 is of the order of unity,

$$q^2 \approx \frac{1}{4} \frac{\omega_\beta^2 r_0^2}{\gamma_0^2 V_0^2} << 1 \quad . \tag{28a}$$

Equivalently

$$q^2 \approx (45 \times 10^{-16} \frac{n_{01} r_0^2}{\gamma_0^3} << 1 \tag{28b}$$

where n_{01} and r_0 are expressed in MKS units. For example if $\gamma_0 \approx 10$ and $r_0 \approx 1$ cm then

$$q^2 \approx (45 \times 10^{-23}) n_{01} << 1 \tag{29}$$

or

$$n_{01} << 2 \times 10^{21}/m^3 = 2 \times 10^{15}/\text{cm}^3 \quad . \tag{30}$$

III THE BEAM AND PLASMA CURRENT

The perturbed beam current response is given by

$$\vec{J}^{B} = -e\gamma_0^5 m^3 \int d^3v(\vec{V}_0 + \vec{v})f_1 \qquad (31)$$

where

$$f_1 = e\int_{\substack{-\infty \\ (ZOO)}}^{t} \left[\vec{E}(\vec{x}',t') + \frac{1}{C}\vec{V}' \times \vec{B}(\vec{x}',t')\right] \cdot \nabla_{p'} f'_{01} dt' \quad . \qquad (32)$$

f_{01} is the equilibrium distribution function, and (ZOO) denotes that the integral is to be done along the zero-order orbits (see Section IV of TI).

A solution of the equilibrium Boltzmann equation for the beam particles which is consistent with the orbit equations, Eqs. (25) and (26), is any arbitrary function of the constants of the motion. That is,

$$f_{01} = f_{01}(v_\theta, v_r^2 - \omega_\beta^2 G^2 \epsilon^2, v_z) \qquad (33)$$

where $\epsilon^2 = r_0^2 - r^2$. However, note that Eqs. (25) and (26) are not consistent with the original Boltzmann equation unless $dV_z/dt \approx 0$. This last condition actually means that γ_0^2 must be very large or, equivalently, V_0 must be very close to c, the speed of light. However, since

$$\frac{v_r^2}{\gamma_0^2 V_0^2} = \frac{\omega_\beta^2 G^2 \epsilon^2}{\gamma_0^2 V_0^2} \leq 4q^2 << 1 \qquad (34a)$$

then the actual variation of v_r with radius is probably not essential and will be approximated in f_{01}, leading to considerable simplification. Accordingly, in f_{01} ϵ^2 is replaced by its average over the cross-section

$$\overline{\epsilon^2} \equiv \rho^2 = \frac{1}{2} r_0^2 \quad . \qquad (34b)$$

So

$$f_{01} = f_{01}(v_\theta, v_r^2 - \omega_\beta^2 G^2 \rho^2, v_z) \quad . \tag{35}$$

In particular f_{01} is taken to be

$$f_{01} \approx \frac{n_{01}\omega_\beta G\rho}{[2\pi\gamma_0^7 m^5 \theta_\parallel]^{1/2}} \delta(v_\theta)\delta(v_r^2 - \omega_\beta^2 G^2\rho^2)\, e^{-\frac{1}{2}\frac{\gamma_0^3 m}{\theta_\parallel} v_z^2} \quad . \tag{36}$$

It should be noted that the full time dependence of $\vec{v}_\perp'$ is retained in the Lorentz force part of Eq. (32).

Let

$$\nabla_p f_{01} = -\vec{S}(\vec{v}) f_{01} \quad . \tag{37a}$$

Then

$$\vec{S}(\vec{v}) = -\frac{1}{\gamma_0 m}\frac{\delta'(v_\theta)}{\delta(v_\theta)}\hat{\theta} - \frac{2\Delta}{\gamma_0 m}\vec{v}_r + \frac{\vec{v}_z}{\theta_\parallel} - \frac{1}{2\gamma_0^3 m V_0}\left(\frac{V_0}{V_0 + v_z} - 2\omega_\beta^2\rho^2\Delta\right)\hat{z} \tag{37b}$$

where

$$\Delta \equiv \frac{\delta'(v_r^2 - \omega_\beta^2 G^2\rho^2)}{\delta(v_r^2 - \omega_\beta^2 G^2 \rho^2)} \quad .$$

The facts that $G^2 = (V_0 + v_z)/V_0$ and $dG^2/dv_z = 1/V_0$ have been used.

Letting

$$\vec{E}(\vec{x}, t) = \vec{E}(\vec{r})e^{i(kz-\omega t)} \tag{38}$$

then

$$f_1 = -e e^{i(kz-\omega t)} \int_{0}^{\infty} \vec{F}(\vec{r}', \vec{v}')e^{i[\Omega - kv_z]\tau}\, d\tau \cdot \vec{S}(\vec{v}) f_{01} \tag{39}$$

(ZOO)

where

$$\vec{F}(\vec{r}', \vec{v}') = \vec{E}(\vec{r}') + \frac{\vec{V}_0 + \vec{v}'}{i\omega} \times (\nabla_\perp' + ik\hat{z}) \times \vec{E}(\vec{r}')$$

and $\Omega = \omega - kV_0 + i\nu_B$ where $\nu_B > 0$.

For the moment the limit $\nu_B \to 0+$ will be taken after the integration is performed.

Expanding $\vec{F}(\vec{r}\,',\vec{v}\,')$ in a Taylor series

$$\vec{F}(\vec{r}\,',\vec{v}\,') = \vec{F}(\vec{r},\vec{v}) + \frac{\partial\vec{F}}{\partial x'}\bigg|_{\tau=0}(x'-x) + \frac{\partial\vec{F}}{\partial y'}\bigg|_{\tau=0}(y'-y)$$

$$+ \frac{\partial\vec{F}}{\partial v_x'}\bigg|_{\tau=0}(v_x'-v_x) + \frac{\partial\vec{F}}{\partial v_y'}\bigg|_{\tau=0}(v_y'-v_y) + \frac{\partial\vec{F}}{\partial v_z'}\bigg|_{\tau=0}(v_z'-v_z) + \ldots \qquad (40)$$

since $\vec{r}\,' = \vec{r}$ and $\vec{v}\,' = \vec{v}$ at $\tau = 0$.

Now, for example,

$$x' - x = x(\cos\omega_0\tau - 1) - \frac{1}{\omega_0} v_x \sin\omega_0\tau$$

and

$$v_x' - v_x = v_x(\cos\omega_0\tau - 1) + \omega_0 x \sin\omega_0\tau$$

so

$$\frac{\partial\vec{F}}{\partial x'}\bigg|_{\tau=0} = \frac{1}{\cos\omega_0\tau}\frac{\partial F}{\partial x}\bigg|_{\tau=0} = \frac{\partial F}{\partial x} \quad .$$

Also

$$v_z' - v_z = 0 \quad .$$

Thus, in compact form,

$$\vec{F}(\vec{r}\,',\vec{v}\,') = \sum_{n=0}^{\infty}\frac{1}{n!}\left[(\vec{r}\cdot\nabla + \vec{v}_\perp\cdot\nabla_v)(\cos\omega_0\tau - 1) + (\omega_0\vec{r}\cdot\nabla_v - \frac{1}{\omega_0}\vec{v}_\perp\cdot\nabla)\sin\omega_0\tau\right]^n \vec{F}(\vec{r},\vec{v}) \qquad (41)$$

where

$$\nabla \equiv \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} \quad \text{and} \quad \nabla_v \equiv \frac{\partial}{\partial v_x}\hat{x} + \frac{\partial}{\partial v_y}\hat{y} \quad .$$

More conveniently, Eq. (41) can be expressed as

$$\vec{F}(\vec{r}\,',\vec{v}\,') = \sum_{n=0}^{\infty} \frac{2^n}{n!} \left(\sin\frac{1}{2}\omega_0\tau\right)^n \left[\left(\omega_0\vec{r}\cdot\nabla_v - \frac{1}{\omega_0}\vec{v}_\perp\cdot\nabla\right)\cos\frac{1}{2}\omega_0\tau\right.$$

$$\left. - (\vec{r}\cdot\nabla + \vec{v}_\perp\cdot\nabla_v)\sin\frac{1}{2}\omega_0\tau\right]^n \vec{F}(\vec{r},\vec{v}) \quad . \tag{42}$$

Defining

$$T_n = \frac{2^{n+1}}{n!\,\omega_0}\int_0^{\infty} (\sin u)^n \left[\left(\omega_0\vec{r}\cdot\nabla_v - \frac{1}{\omega_0}\vec{v}_\perp\cdot\nabla\right)\cos u\right.$$

$$\left. - (\vec{r}\cdot\nabla + \vec{v}_\perp\cdot\nabla_v)\sin u\right]^n e^{Du}du$$

where

$$D = i\,\frac{2(\Omega - kv_z)}{\omega_0} \quad ,$$

then

$$f_1 = -ee^{i(kz-\omega t)}\sum_{n=0}^{\infty}[T_n\vec{F}(\vec{r},\vec{v})]\cdot\vec{S}(\vec{v})f_{01} \tag{43}$$

and

$$J_a^B = e^2\gamma_0^5 m^3 e^{i(kz-\omega t)}\int d^3v(\vec{V}_0 + \vec{v})_a \sum_{n=0}^{\infty}[T_n\vec{F}(\vec{r},\vec{v})]\cdot\vec{S}(\vec{v})f_{01} \quad . \tag{44}$$

Letting

$$j_{an}^B = \int d^3v(\vec{V}_0 + \vec{v})_a [T_n\vec{F}(\vec{r},\vec{v})]\cdot\vec{S}(\vec{v})f_{01} \tag{45}$$

then [omitting the factor $e^{i(kz-\omega t)}$ from now on]

$$-4\pi i\omega J_a^B = -\omega_\perp^2\frac{\gamma_0^5 m^3}{n_{01}}\,i\,\omega\gamma_0 m\sum_{n=0}^{\infty} j_{an}^B \quad . \tag{46}$$

Evaluation of j_{a0}^B and j_{a1}^B is straightforward though lengthy. The calculation is therefore only outlined here.

T_0 and T_1 are easily found to be

$$T_0 = \frac{i}{\Omega - kv_z} \tag{47a}$$

and

$$T_1 = \frac{\omega_0}{\omega_0^2 - (\Omega - kv_z)^2}\left[\omega_0 \vec{r}\cdot\nabla_v - \frac{1}{\omega_0}\vec{v}_\perp\cdot\nabla - i\,\frac{\omega_0}{\Omega - kv_z}\,(\vec{r}\cdot\nabla + \vec{v}_\perp\cdot\nabla_v)\right] \quad . \tag{47b}$$

The integrals over $\vec{v}$ are carried out using

$$\begin{aligned} \int d^3v v_r^n f_{01} &= (\omega_0\rho)^n \int d^3v f_{01} && \text{if } n \text{ is even} \\ &= 0 && \text{if } n \text{ is odd} \end{aligned} \tag{48a}$$

$$\begin{aligned} d^3v v_\theta^n f_{01} &= \int d^3v f_{01} && \text{if } n = 0 \\ &= 0 && \text{if } n \neq 0 \end{aligned} \tag{48b}$$

$$\begin{aligned} \int d^3v v_r^n \Delta f_{01} &= \frac{1-n}{2}(\omega_0\rho)^{n-2}\int d^3v f_{01} && \text{if } n \text{ is even} \\ &= 0 && \text{if } n \text{ is odd} \end{aligned} \tag{48c}$$

$$\begin{aligned} \int d^3v v_\theta^n \frac{\delta'(v_\theta)}{\delta(v_\theta)} f_{01} &= -\int d^3v f_{01} && \text{if } n = 1 \\ &= 0 && \text{if } n \neq 1 \quad . \end{aligned} \tag{48d}$$

The required vector identities, expressed in cylindrical coordinates for arbitrary normal mode, are given in the appendix. That is, the electric field is expanded as

$$\vec{E}(\vec{r}) = \sum_{\ell=0}^{\infty} \vec{E}_\ell(r)e^{i\ell\theta}$$

(the subscript, ℓ, will be omitted below).

The perturbed beam current, including contributions arising only from the first two terms of the Taylor series, for normal mode ℓ is

$$-4\pi i\omega J_r^B = B_1 E_r + (B_3 + b_3)E_z' + b_1(ikE_r - E_z')$$

$$+ b_2 r E_z'' + b_4\left[\frac{r}{\rho^2} E_r' + \frac{\ell^2}{r^2} E_r + \frac{i\ell}{r}\left(E_\theta' + \frac{1}{r} E_\theta\right)\right]$$

$$+ b_5 r\left[E_z'' + \frac{1}{r} E_z' - ik\left(E_r' + \frac{1}{r} E_r\right)\right] - \omega_\perp^2 \frac{\gamma_0^5 m^3}{n_{01}} i\omega\gamma_0 m \sum_{n=2}^{\infty} j_{rn}^B$$

$$-4\pi i\omega J_\theta^B = B_1 E_\theta + B_3 \frac{i\ell}{r} E_z + b_2 i\ell\left(E_z' - \frac{1}{r} E_z\right)$$

$$+ b_4\left[E_\theta'' + \frac{1}{r} E_\theta' - \frac{1}{r^2} E_\theta - \frac{i\ell}{r}\left(E_r' - \frac{1}{r} E_r\right) - \frac{1}{\rho^2}(E_\theta - i\ell E_r)\right]$$

$$- \omega_\perp^2 \frac{\gamma_0^5 m^3}{n_{01}} i\omega\gamma_0 m \sum_{n=2}^{\infty} j_{\theta n}^B \tag{49}$$

and

$$-4\pi i\omega J_z^B = B_2 E_z + (B_3 + b_2)\left(E_r' + \frac{1}{r} E_r + \frac{i\ell}{r} E_\theta\right)$$

$$+ (B_4 + b_6)\left(E_z'' + \frac{1}{r} E_z' - \frac{\ell^2}{r^2} E_z\right) + b_7 r E_r + b_8 r E_z'$$

$$+ b_9(ikE_r' - E_z'') - \omega_\parallel^2 \frac{\gamma_0^5 m^3}{n_{01}} i\omega\gamma_0^3 m \sum_{n=2}^{\infty} j_{zn}^B$$

where

$$B_1 = \omega_\perp^2$$

$$B_2 = \omega_\parallel^2 \frac{\gamma_0^5 m^3}{n_{01}} \frac{\gamma_0^3 m}{\theta_\parallel} \omega \int d^3v \, \frac{(V_0 + v_z)v_z}{\Omega - kv_z} f_{01}$$

$$B_3 = -\omega_\perp^2 \frac{\gamma_{0m}^5{}^3}{n_{01}} i \int d^3v \frac{V_0 + v_z}{\Omega - kv_z} f_{01}$$

$$B_4 = -\omega_\perp^2 \frac{\gamma_0^5 m^3}{n_{01}} \int d^3v \frac{(V_0 + v_z)^2}{(\Omega - kv_z)^2} f_{01}$$

$$b_1 = -\omega_\perp^2 \frac{\gamma_0^5 m^3}{n_{01}} \frac{iV_0}{u_B^2} \frac{\omega_\beta^2 \rho^2}{\gamma_0^2 V_0^2} \int d^3v \frac{v_z(V_0 + v_z) - u_B^2}{\Omega - kv_z} f_{01}$$

$$b_2 = \omega_\perp^2 \frac{\gamma_0^5 m^3}{n_{01}} \frac{\omega_\beta^2}{V_0} i \int d^3v \frac{V_0 + v_z}{\Omega - kv_z} \cdot \frac{V_0 + v_z}{\omega_\beta^2 G^2 - (\Omega - kv_z)^2} f_{01}$$

$$b_3 = \omega_\perp^2 \frac{\gamma_0^5 m^3}{n_{01}} \frac{i\omega V_0}{u_B^2} \frac{\omega_\beta^2 \rho^2}{\gamma_0^2 V_0^2} \int d^3v \frac{v_z(V_0 + v_z) - u_\beta^2}{\omega_\beta^2 G^2 - (\Omega - kv_z)^2} f_{01}$$

$$b_4 = -\omega_\perp^2 \frac{\gamma_0^5 m^3}{n_{01}} \frac{\omega_\beta^2 \rho^2}{V_0} \int d^3v \frac{V_0 + v_z}{\omega_\beta^2 G^2 - (\Omega - kv_z)^2} f_{01}$$

$$b_5 = -\omega_\perp^2 \frac{\gamma_0^5 m^3}{n_{01}} i \frac{\omega_\beta^2}{u_B^2} \frac{\omega_\beta^2 \rho^2}{\gamma_0^2 V_0^2} \int d^3v \frac{V_0 + v_z}{\Omega - kv_z} \cdot \frac{v_z(V_0 + v_z) - u_B^2}{\omega_\beta^2 G^2 - (\Omega - kv_z)^2} f_{01}$$

$$b_6 = \omega_\perp^2 \frac{\gamma_0^5 m^3}{n_{01}} \frac{\omega_\beta^2}{V_0} \int d^3v \frac{(V_0 + v_z)^2}{(\Omega - kv_z)^2} \frac{V_0 + v_z}{\omega_\beta^2 G^2 - (\Omega - kv_z)^2} f_{01}$$

$$b_7 = -\omega_\parallel^2 \frac{\gamma_0^5 m^3}{n_{01}} i \frac{\omega_\beta^2}{V_0 u_B^2} \int d^3v \frac{kv_z(V_0 + v_z)^2}{\omega_\beta^2 G^2 - (\Omega - kv_z)^2} f_{01}$$

$$b_8 = -\omega_\parallel^2 \frac{\gamma_0^5 m^3}{n_{01}} \frac{\omega_\beta^2}{V_0 u_B^2} \int d^3v \frac{V_0 + v_z}{\Omega - kv_z} \cdot \frac{kv_z(V_0 + v_z)^2}{\omega_\beta^2 G^2 - (\Omega - kv_z)^2} f_{01}$$

$$b_9 = \omega_\perp^2 \frac{\gamma_0^5 m^3}{n_{01}} \frac{V_0}{u_B^2} \frac{\omega_\beta^2 \rho^2}{\gamma_0^2 V_0^2} \int d^3v \frac{(V_0 + v_z)[v_z(V_0 + v_z) - U_B^2]}{\omega_\beta^2 G^2 - (\Omega - kv_z)^2} f_{01}$$

where

$$\omega_{\perp}^2 = \frac{4\pi e^2 n_{01}}{\gamma_0 m}$$

$$\omega_{\parallel}^2 = \frac{4\pi e^2 n_{01}}{\gamma_0^3 m}$$

and

$$\theta_{\parallel} = \gamma_0^3 m u_B^2 \quad .$$

Now it should be noted that b_1, b_3, b_5, and b_9 are proportional to $(\omega_\beta^2 \rho^2)/(\gamma_0^2 V_0^2)$, which is of the order of q^2. Since the use of the orbit equations, Eqs. (25) and (26), presupposes the neglect of terms of order q^2, b_1, b_3, b_5, and b_9 should be omitted for consistency. However, for ω_β^2 larger or of the same magnitude as Ω^2, the remaining b_i's give contributions that cannot be neglected. Indeed, if $\omega_\beta^2 >> \Omega^2$, $b_2 \approx -B_3$, for example. Hence for $\omega_\beta^2 \approx \Omega^2$ or $\omega_\beta^2 >> \Omega^2$ the method used here to evaluate Eq. (39) is inapplicable since the efficacy of the Taylor series expansion depends completely upon being able to cut off the series after a very few terms. However, in the limit $\omega_\beta^2 << \Omega^2$ all the b_i's tend to zero and only the B_i's are left. It is shown now that the remaining terms in the Taylor expansion (*i.e.*, $j_{\alpha n}^B$ for $n \geq 2$) all vanish as $\omega_\beta^2/\Omega^2 \to 0$.

For $n = 2$

$$j_{\alpha 2}^B = \int d^3v (\vec{V}_0 + \vec{v})_\alpha [T_2 \vec{F}(\vec{r},\vec{v})] \cdot \vec{S}(\vec{v}) f_{01} \quad . \tag{50}$$

Now

$$T_2 = A_1 I_1 + A_2 I_2 + A_3 I_3 \tag{51}$$

where

$$A_1 = \left(\omega_0 \vec{r} \cdot \nabla_v - \frac{1}{\omega_0} \vec{v}_\perp \cdot \nabla\right)^2$$

$$A_2 = -\left(\omega_0 \vec{r} \cdot \nabla_v - \frac{1}{\omega_0} \vec{v}_\perp \cdot \nabla\right)(\vec{r} \cdot \nabla + \vec{v}_\perp \cdot \nabla_v)$$

$$- (\vec{r} \cdot \nabla + \vec{v}_\perp \cdot \nabla_v)\left(\omega_0 \vec{r} \cdot \nabla_v - \frac{1}{\omega_0} \vec{v}_\perp \cdot \nabla\right)$$

$$A_3 = (\vec{r} \cdot \nabla + \vec{v}_\perp \cdot \nabla_v)^2$$

and,

$$D = i\frac{2(\Omega - kv_z)}{\omega_0} ,$$

$$I_1 = \frac{4}{\omega_0}\int_0^\infty (\sin u)^2 (\cos u)^2 e^{Du} du$$

$$I_2 = \frac{4}{\omega_0}\int_0^\infty (\sin u)^3 \cos u e^{Du} du$$

$$I_3 = \frac{4}{\omega_0}\int_0^\infty (\sin u)^4 e^{Du} du .$$

Neglecting the possibility of accidental cancellations $A_i\vec{F}(\vec{r},\vec{v})$ can be written in terms of powers of $v_\perp$, where $v_\perp$ signifies either v_r or v_θ. Since, for example,

$$(\vec{r} \cdot \nabla_v)^2\vec{F} = 0$$

$$(\vec{v}_\perp \cdot \nabla)^2\vec{F} = \vec{\alpha}_1 v_\perp^3 + \vec{\alpha}_2 v_\perp^2$$

$$(\vec{r} \cdot \nabla_v)(\vec{v}_\perp \cdot \nabla)\vec{F} = \vec{\alpha}_3 + \vec{\alpha}_4 v_\perp , \quad \text{etc.}$$

where the $\vec{\alpha}_i$'s are vectors independent of the transverse velocities and also independent of ω_0 ,

$$A_1\vec{F} = \vec{\beta}_1 + \vec{\beta}_2 v_\perp + \frac{1}{\omega_0^2}(\vec{\beta}_3 v_\perp^2 + \vec{\beta}_4 v_\perp^3)$$

$$A_2\vec{F} = \omega_0\vec{\beta}_5 + \frac{1}{\omega_0}(\vec{\beta}_6 v_\perp + \vec{\beta}_7 v_\perp^2)$$

$$A_3F = \vec{\beta}_8 v_\perp \tag{52}$$

where $\vec{\beta}_i$'s are linear combinations of the $\vec{\alpha}_i$'s.

Consider j^B_{z2}:

$$j^B_{z2} = \int d^3v(V_0 + v_z)\left\{I_1\left[\vec{\beta}_1 + \vec{\beta}_2 v_\perp + \frac{1}{\omega_0^2}(\vec{\beta}_3 v_\perp^2 + \vec{\beta}_4 v_\perp^3)\right] + I_2\left[\omega_0\vec{\beta}_5 + \frac{1}{\omega_0}(\vec{\beta}_6 v_\perp + \vec{\beta}_7 v_\perp^2)\right] + I_3\vec{\beta}_8 v_\perp\right\}\cdot\vec{S}(\vec{v})f_{01} \quad . \qquad (53)$$

Using the explicit expression for $\vec{S}(\vec{v})$ and Eqs. (48a-d) the general expression for j^B_{z2} is

$$j^B_{z2} = \int d^3v(V_0 + v_z)\left\{I_1\left[\gamma_1 + \gamma_2 + \frac{1}{\omega_0^2}\gamma_3\omega_0^2 + \frac{1}{\omega_0^2}(\gamma_4\omega_0^4 + \gamma_4^1\omega_0^2)\right] + I_2\left[\omega_0\gamma_5 + \frac{1}{\omega_0}\gamma_6 + \frac{1}{\omega_0}\gamma_7\omega_0^2\right] + I_3\gamma_8\right\}f_{01}$$

$$= \int d^3(V_0 + v_z)\left\{I_1[\Gamma_1 + \omega_0^2\Gamma_2] + I_2\left[\frac{1}{\omega_0}\Gamma_3 + \omega_0\Gamma_4\right] + I_3\Gamma_5\right\}f_{01} \qquad (54)$$

where the γ_i's, and therefore the Γ_i's, are independent of $v_\perp$ and ω_0.

Now

$$I_1 = -\frac{8}{\omega_0 D}\frac{1}{16 + D^2} \xrightarrow[\omega_0\to 0]{} -\frac{8}{\omega_0 D^3} = -\frac{i}{\omega_0}\left(\frac{\omega_0}{\Omega - kv_z}\right)^3$$

$$I_2 = \frac{24}{\omega_0(16 + D^2)(4 + D^2)} \xrightarrow[\omega_0\to 0]{} \frac{24}{\omega_0 D^4} = \frac{3}{2\omega_0}\left(\frac{\omega_0}{\Omega - kv_z}\right)^4$$

$$I_3 = -\frac{96}{\omega_0 D(16 + D^2)(4 + D^2)} \xrightarrow[\omega_0\to 0]{} -\frac{96}{\omega_0 D^5} = \frac{3i}{\omega_0}\left(\frac{\omega_0}{\Omega - kv_z}\right)^5 \quad . \qquad (55)$$

So finally

$$j^B_{z2} \xrightarrow[\omega_0 \to 0]{} \int d^3v(V_0 + v_z)\left[\left(\frac{\omega_0}{\Omega - kv_z}\right)^2 \Sigma_1(v_z) + \left(\frac{\omega_0}{\Omega - kv_z}\right)^4 \Sigma_2(v_z)\right] f_{01} \quad (56)$$

where Σ_1 and Σ_2 are independent of ω_0 .

Hence j^B_{z2} is negligible for $\omega_\beta^2 << \Omega^2$. The same result immediately follows for $\vec{j}^B_{\perp 2}$ since the extra factor of $\vec{v}_\perp$ in its integrand requires it to go to zero when $\omega_0 \to 0$ as fast or faster than j^B_{z2}. It is quite obvious from the form of T_n that the same result holds for $j^B_{\alpha n}$ for $n > 2$.

Thus in the limit of $\Omega^2 >> \omega_\beta^2$, the complete beam current is found from the first two terms of the Taylor series. Moreover the result is exactly the same as that derived in TI using straight-line orbits, generalized here to arbitrary normal mode.

Omitting calculational details, the beam current is now given for the case that ν_B can be regarded as the average beam collision frequency and is not negligible relative to $\Omega^2 >> \omega_\beta^2$:

$$-4\pi i\omega J^B_r = B_1 E_r + B_3 E'_z$$

$$-4\pi i\omega J^B_\theta = B_1 E_\theta + B_3 \frac{i\ell}{r} E_z$$

$$-4\pi i\omega J^B_z = B_z E_z + B_3\left(E'_r + \frac{1}{r}E_r + \frac{i\ell}{r}E_\theta\right) + B_4\left(E''_z + \frac{1}{r}E'_z - \frac{\ell^2}{r^2}E_z\right) \quad (57)$$

where ($\Omega \equiv \omega - kV_0$ from now on)

$$B_1 = \omega_\perp^2 \frac{\Omega}{\Omega + i\nu_B}\left[1 + \frac{kV_0}{\Omega}\frac{\gamma_0^5 m^3}{n_{01}}\int d^3v \frac{v_z}{V_0}\frac{\Omega - kv_z}{\Omega + i\nu_B - kv_z} f_{01}\right]$$

$$B_2 = \omega_\parallel^2 \frac{\gamma_0^5 m^3}{n_{01}}\frac{\gamma_0^3 m}{\theta_\parallel}\,\omega \int d^3v \frac{v_z(V_0 + v_z)}{\Omega + i\nu_B - kv_z} f_{01}$$

$$B_3 = -\omega_\perp^2 \frac{\gamma_0^5 m^3}{n_{01}} i \int d^3v \frac{V_0 + v_z}{\Omega + i\nu_B - kv_z} f_{01}$$

and

$$B_4 = -\omega_\perp^2 \frac{\gamma_0^5 m^3}{n_{01}} \int d^3v \frac{(V_0 + v_z)^2}{(\Omega + i\nu_B - kv_z)^2} f_{01} \quad .$$

The plasma current is now calculated for arbitrary normal mode assuming straight-line orbits for both the electrons and ions. That is, the effect upon the plasma of the beam's magnetic field is neglected. This places a restriction upon the results for the plasma current which can be approximately obtained from the hydrodynamical treatment of Mjolsness, Enoch, and Langmuir.[4] They find that for a cold beam and plasma, the plasma current can be written as (neglecting the ionic contribution)

$$-4\pi i\omega J_r^P = \omega_e^2 \frac{E_r + \mu(r)E_z}{1 + \mu^2(r)}$$

$$-4\pi i\omega J_\theta^P = \omega_e^2 E_\theta$$

$$-4\pi i\omega J_z^P = \omega_e^2 \frac{E_z - \mu(r)E_r}{1 + \mu^2(r)} \tag{58}$$

where

$$\omega_e^2 = \frac{4\pi e^2 n_{02}}{m} \frac{\omega}{\omega + i\nu_e}$$

and

$$\mu(r) = \frac{i}{\omega + i\nu_e} \frac{e}{mc} B_0(r) \quad ,$$

ν_e being the collision frequency of the plasma electrons.

For a uniform beam of radius r_0

$$\mu(r) = -\frac{i}{\omega + i\nu_e} \frac{\gamma_0 \omega_\beta^2}{V_0} f(r) \tag{59}$$

where

$$f(r) = \begin{cases} r & r < r_0 \\ \dfrac{r_0^2}{r} & r > r_0 \end{cases} .$$

Hence neglecting the effect of $\vec{B}_0(r)$ on the plasma is equivalent to the restriction

$$|\mu(r)| \ll 1 \quad . \tag{60}$$

This inequality is satisfied for all r if

$$\frac{|\omega + i\nu_e|}{\omega_\beta} \gg \frac{\gamma_0 \omega_\beta r_0}{V_0} \tag{61}$$

or, equivalently,

$$\frac{|\omega + i\nu_e|}{\omega_\beta} \gg 2\gamma_0^2 q \tag{62}$$

where

$$q^2 \approx \frac{1}{4} \frac{\omega_\beta^2 r_0^2}{\gamma_0^2 \nu_0^2} \quad .$$

(It should be noted however that Mjolsness, *et al.*,[4] find that the effect on their dispersion relation is negligible even for large values of $\mu(r)$. Therefore, Inequality (62) should probably not be regarded as being as essential to the accuracy of the analysis as is, for example, the high frequency criterion, $|\omega - kV_0|^2 \gg \omega_\beta^2$).

Under the restriction (62), the plasma current is calculated for arbitrary normal mode using the same methods as in TI. Relevant vector identities are given in the appendix. The result is

$$-4\pi i\omega J_r^P = (P_1 + P_2)E_r + P_3 E_z'$$

$$-4\pi i\omega J_\theta^P = (P_1 + P_2)E_\theta + P_3 \frac{i\ell}{r} E_z$$

$$-4\pi i\omega J_z^P = (P_1 + P_4)E_z + P_3\left(E_r' + \frac{1}{r}E_r + \frac{i\ell}{r}E_\theta\right)$$

$$+ P_5\left(E_z'' + \frac{1}{r}E_z' - \frac{ik}{r}E_r - \frac{\ell^2}{r^2}E_z\right)$$

$$+ P_6\left(E_z'' - \frac{\ell^2}{r^2}E_z\right) \qquad (63)$$

where

$$P_1 = \omega_P^2$$

$$P_2 = \int d^3v \frac{kv_z}{\omega + i\nu_p - kv_z} f_{23}$$

$$P_3 = -i(\omega + i\nu_\rho) \int d^3v \frac{v_z}{(\omega + i\nu_\rho - kv_z)^2} f_{23}$$

$$P_4 = \int d^3v \frac{kv_z^3}{\omega + i\nu_\rho - kv_z}\left[\omega_e^2 \frac{m^3}{n_{02}} \frac{m}{kT_e} f_{02} + \omega_i^2 \frac{M^3}{n_{03}} \frac{M}{KT_i} f_{03}\right]$$

$$P_5 = -\int d^3v \frac{v_z^2}{(\omega + i\nu_\rho - kv_z)^2} f_{23}$$

$$P_6 = -\int d^3v \frac{kv_z^3}{(\omega + i\nu_\rho - kv_z)^3} f_{23} \quad .$$

(Note that $P_2 = ikP_3 + k^2P_5$).

Here ν_ρ is meant to denote ν_e when integrating over the electron velocity distribution and ν_i when integrating over the ions,

$$f_{23} = \omega_e^2 \frac{m^3}{n_{02}} f_{02} + \omega_i^2 \frac{M^3}{n_{03}} f_{03}$$

and

$$\omega_\rho^2 = \omega_e^2 + \omega_i^2$$

where

$$\omega_e^2 = \frac{4\pi e^2 n_{02}}{m} \frac{\omega}{\omega + i\nu_e} \quad \text{and} \quad \omega_i^2 = \frac{4\pi e^2 n_{03}}{M} \frac{\omega}{\omega + i\nu_i} .$$

The total perturbed current for arbitrary normal mode can be written as

$$-4\pi i\omega J_r = \sigma_1^2 E_r + \sigma_2^2 \frac{1}{ik} E_z'$$

$$-4\pi i\omega J_\theta = \sigma_1^2 E_\theta + \sigma_2^2 \frac{1}{ik} \frac{i\ell}{r} E_z$$

$$-4\pi i\omega J_z = \sigma_3^2 E_z + \sigma_2^2 \frac{1}{ik} \left(E_r' + \frac{i\ell}{r} E_\theta\right)$$

$$+ \sigma_4^2 \frac{1}{ik}\frac{1}{r} E_r - \sigma_5^2 \frac{1}{k^2} \left(E_z'' + \frac{1}{r} E_z' - \frac{\ell^2}{r^2} E_z\right)$$

$$- \sigma_6^2 \frac{1}{k^2} \left(E_z'' - \frac{\ell^2}{r^2} E_z\right) \qquad (64)$$

where

$$\sigma_1^2 = B_1 + P_1 + P_2$$

$$\sigma_2^2 = ik(B_3 + P_3)$$

$$\sigma_3^2 = B_2 + P_1 + P_4$$

$$\sigma_4^2 = ikB_3 + P_2$$

$$\sigma_5^2 = -k^2(B_4 + P_5)$$

$$\sigma_6^2 = -k^2 P_6 .$$

In terms of the usual dispersion integral

$$Z_j(s_j) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, \frac{e^{-\frac{1}{2}t^2}}{t - s_j - i\epsilon} = \left[i\sqrt{\frac{\pi}{2}} - \int_0^{s_j} dt\, e^{\frac{1}{2}t^2}\right] e^{-\frac{1}{2}s_j^2} \qquad (65)$$

where

$$s_B = \frac{\Omega + i\nu_B}{ku_B} \qquad \text{(for the beam)}$$

$$s_e = \frac{\omega + i\nu_e}{ku_e} \qquad \text{(for the plasma electrons)}$$

$$s_i = \frac{\omega + i\nu_i}{ku_i} \qquad \text{(for the plasma ions)} \quad ,$$

and where

$$\theta_{\|} = \gamma_0^3 m u_B^2 \qquad kT_e = mu_e^2 \qquad KT_i = Mu_i^2 \quad ,$$

$$\sigma_1^2 = \omega_\perp^2 \frac{\Omega}{\Omega + i\nu_B}\left[1 + \frac{i\nu_B}{\Omega}(1 + s_B Z_B)\right] + \omega_\rho^2 - \omega_e^2[1 + s_e Z_e] - \omega_i^2[1 + s_i Z_i]$$

$$\sigma_2^2 = -\omega_\perp^2\left[1 + \frac{\omega + i\nu_B}{ku_B} Z_B\right] + \omega_e^2 s_e[Z_e + s_e Z_e'] + \omega_i^2 s_i[Z_i + s_i Z_i']$$

$$\sigma_3^2 = -\omega_{\|}^2 \frac{\omega(\omega + i\nu_B)}{k^2 u_B^2}[1 + s_B Z_B] + \omega_\rho^2 - \omega_e^2[1 + s_e^2 + s_e^3 Z_e] - \omega_i^2[1 + s_i^2 + s_i^3 Z_i]$$

$$\sigma_4^2 = -\omega_\perp^2\left[1 + \frac{\omega + i\nu_B}{ku_B} Z_B\right] - \omega_e^2[1 + s_e Z_e] - \omega_i^2[1 + s_i Z_i] \qquad (66)$$

$$\sigma_5^2 = \omega_\perp^2 \left[1 + 2\,\frac{\omega + i\nu_B}{ku_B}\, Z_B + \frac{(\omega + i\nu_B)^2}{k^2 u_B^2}\, Z_B' \right] + \omega_e^2 [1 + 2s_e Z_e + s_e^2 Z_e']$$

$$+ \omega_i^2 [1 + 2s_i Z_i + s_i^2 Z_i']$$

$$\sigma_6^2 = -\omega_e^2 \left[1 + 3s_e Z_e + 3s_e^2 Z_e' + \frac{1}{2} s_e^3 Z_e'' \right] - \omega_i^2 \left[1 + 3s_i Z_i + 3s_i^2 Z_i' + \frac{1}{2} s_i^3 Z_i'' \right] \quad . \qquad (66)$$

(contd)

Using the low temperature expansion ($|s| > 1$) of $Z(s)$ the cold beam and cold plasma limit is

$$\sigma_1^2 = \omega_\perp^2 \frac{\Omega}{\Omega + i\nu_B} + \omega_\rho^2$$

$$\sigma_2^2 = \omega_\perp^2 \frac{kV_0}{\omega + i\nu_B}$$

$$\sigma_3^2 = \omega_\parallel^2 \frac{\omega(\omega + i\nu_B)}{(\Omega + i\nu_B)^2} + \omega_\rho^2$$

$$\sigma_4^2 = \omega_\perp^2 \frac{kV_0}{\omega + i\nu_B}$$

$$\sigma_5^2 = \omega_\perp^2 \frac{k^2 V_0^2}{(\Omega + i\nu_B)^2}$$

$$\sigma_6^2 = 0 \quad . \qquad (67)$$

IV MAXWELL'S EQUATIONS

The electric and magnetic fields are determined by

$$\nabla \times (\nabla \times \vec{E}) + \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{4\pi}{c^2}\frac{\partial \vec{J}}{\partial t} \tag{68a}$$

and

$$\frac{\partial \vec{B}}{\partial t} = -c\nabla \times \vec{E} \quad . \tag{68b}$$

Defining

$$\vec{E}(\vec{r},z,t) = \sum_{l=0}^{\infty} \vec{E}_l(\vec{r})e^{i(kz-\omega t+l\theta)} \quad , \text{ etc.,} \tag{69}$$

then Eqs. (68a) and (68b) become, using Eq. (64) for $\vec{J}$, (and suppressing the subscript ℓ),

$$[k^2c^2 + \sigma_5^2 + \sigma_6^2]E_z'' + [k^2c^2 + \sigma_5^2]\frac{1}{r}E_z' + \Bigg[k^2(\omega^2 - \sigma_3^2)$$

$$- (k^2c^2 + \sigma_5^2 + \sigma_6^2)\frac{\ell^2}{r^2}\Bigg]E_z = ik[k^2c^2 - \sigma_2^2]\left(E_r' + \frac{i\ell}{r}E_\theta\right)$$

$$+ ik[k^2c^2 - \sigma_4^2]\frac{1}{r}E_r \quad , \tag{70a}$$

$$E_\theta'' + \frac{1}{r}E_\theta' + \left[\frac{\omega^2 - k^2c^2 - \sigma_1^2}{c^2} - \frac{1}{r^2}\right]E_\theta$$

$$= \frac{i\ell}{r}\left(E_r' - \frac{1}{r}E_r\right) + ik\left[1 - \frac{\sigma_2^2}{k^2c^2}\right]\frac{i\ell}{r}E_z \tag{70b}$$

$$\left[\frac{\omega^2 - k^2c^2 - \sigma_1^2}{c^2} - \frac{\ell^2}{r^2}\right]E_r = \frac{i\ell}{r}\left(E_\theta' + \frac{1}{r}E_\theta\right) + ik\left[1 - \frac{\sigma_2^2}{k^2c^2}\right]E_z' \tag{70c}$$

and

$$\frac{i\omega}{c}\vec{B} = \left(\frac{i\ell}{r}E_z - ikE_\theta\right)\hat{r} + (ikE_r - E_z')\hat{\theta} + \left(E_\theta' + \frac{1}{r}E_\theta - \frac{i\ell}{r}E_r\right)\hat{z} \quad . \tag{70d}$$

Equations (70a-c) can be put in a somewhat more compact form by using Eqs. (70b) and (70c) to find E_θ in terms of E_r and E_z and then substituting into Eq. (70a). Also, E_z can be eliminated from Eqs. (70b) and (70c). This all leads to the following set of equations, which are equivalent to Eqs. (70a-c),

$$E_z'' + \frac{2\zeta_1 + 1}{r}E_z' + \left[\beta^2 - \frac{\ell^2}{r^2}\right]E_z = \frac{2ik\zeta_2}{r}E_r \tag{71a}$$

$$E_\theta''' + \frac{2}{r}E_\theta'' + \left[\eta^2 - \frac{\ell^2+1}{r^2}\right]E_\theta' + \left[\eta^2 - \frac{\ell^2-1}{r^2}\right]\frac{1}{r}E_\theta$$

$$= \frac{i\ell}{r}\left\{E_r'' - \frac{1}{r}E_r' + \left[\eta^2 - \frac{\ell^2-1}{r^2}\right]E_r\right\} \tag{71b}$$

$$-\frac{i\ell}{r}E_\theta = E_r' + \frac{1}{r}E_r + \frac{k^2c^2-\sigma_2^2}{ikc^2\eta^2}\left(E_z'' + \frac{1}{r}E_z' - \frac{\ell^2}{r^2}E_z\right) \tag{71c}$$

where

$$c^2\eta^2 = \omega^2 - k^2c^2 - \sigma_1^2$$

$$\beta^2 = k^2\,\frac{(\omega^2-\sigma_3^2)(\omega^2-k^2c^2-\sigma_1^2)}{(\omega^2-k^2c^2-\sigma_1^2)(k^2c^2+\sigma_5^2+\sigma_6^2)+(k^2c^2-\sigma_2^2)^2}$$

$$\zeta_1 = -\frac{1}{2}\,\frac{\sigma_6^2(\omega^2-k^2c^2-\sigma_1^2)}{(\omega^2-k^2c^2-\sigma_1^2)(k^2c^2+\sigma_5^2+\sigma_6^2)+(k^2c^2-\sigma_2^2)^2}$$

$$\zeta_2 = +\frac{1}{2}\frac{(\sigma_2^2 - \sigma_4^2)(\omega^2 - k^2c^2 - \sigma_1^2)}{(\omega^2 - k^2c^2 - \sigma_1^2)(k^2c^2 + \sigma_5^2 + \sigma_6^2) + (k^2c^2 - \sigma_2^2)^2} .$$

(Both ζ_1 and ζ_2 equal zero for zero plasma temperature since $\sigma_2^2 - \sigma_4^2 = ikP_3 - P_2 = -k^2P_5$ and $\sigma_6^2 = -k^2P_6$.)

For $\ell = 0$ Eqs. (71a-c) easily decouple. Exact solutions and the resultant dispersion relations for this normal mode are given in the next section. When the plasma temperature is zero the $\ell \neq 0$ modes can also be uncoupled and this will be treated in detail in Section VI. In general, however, for $\ell \neq 0$ and non-zero temperature uncoupling of Eqs. (71a-c) results in a sixth order differential equation for $E_z(r)$. This equation can be written as [defining $d^nE_z(r)/dr^n \equiv E_z(r)$ for $n = 0$]

$$\sum_{n=0}^{6} \frac{a_n(r)}{r^{6-n}} \frac{d^nE_z(r)}{dr^n} = 0 \qquad (72)$$

where

$$a_6 = 1$$

$$a_5 = 11 + 2\zeta$$

$$a_4(r) = [\beta^2 + \eta^2]r^2 - [3\ell^2 - 29 - 12\zeta]$$

$$a_3(r) = 10\beta^2r^2 + [6 + 2\zeta]\eta^2r^2 - [10\ell^2 - 14 + (2\ell^2 - 5)2\zeta]$$

$$a_2(r) = \eta^2\beta^2r^4 + [23 - 2\ell^2]\beta^2r^2 + [7 - 2\ell^2 + 6\zeta]\eta^2r^2 - [5\ell^2 - 3\ell^4 + 1 + (2\ell^2 + 1)2\zeta]$$

$$a_1(r) = 5\eta^2\beta^2r^4 + [9 - 6\ell^2]\beta^2r^2 + [1 - 2\ell^2 + 2\zeta - 2\zeta\ell^2]\eta^2r^2 - [\ell^4 + 3\ell^2 - 1 - 2\zeta(\ell^2 - 1)^2]$$

$$a_0(r) = (4 - \ell^2)\eta^2\beta^2r^4 + \ell^2(\ell^2 - 4)\beta^2r^2 + \ell^4\eta^2r^2 - \ell^4(\ell^2 - 4)$$

and

$$\zeta = \zeta_1 + \frac{k^2c^2 - \sigma_2^2}{c^2\eta^2}\zeta_2 .$$

Solving Eq. (72) for $E_z(r)$, $E_r(r)$ can be immediately obtained from Eq. (71a) and $E_\theta(r)$ can then be obtained from Eq. (71c).

The dispersion relations are obtained by satisfying the proper boundary conditions at the beam-plasma interface and at any other boundary present. The derivation of the correct boundary conditions for $\ell \neq 0$ follows the same method as used in TI for $\ell = 0$. For a particular mode ℓ, let $\sigma^*\delta(r - r_0)$ and $\vec{J}^*\delta(r - r_0)$ represent the surface charge and current, respectively, on the boundary located at $r = r_0$, where $(r - r_0)$ is the one dimensional Dirac delta function. Integrating Maxwell's equations across the boundary gives

$$\Delta E_r = 4\pi\sigma^* \tag{73a}$$

$$\Delta E_\theta = 0 \tag{73b}$$

$$\Delta E_z = 0 \tag{73c}$$

$$\Delta B_r = 0 \tag{73d}$$

$$\Delta B_\theta = \frac{4\pi}{c} J_z^* \tag{73e}$$

$$\Delta B_z = -\frac{4\pi}{c} J_\theta^* \tag{73f}$$

and

$$\frac{i\ell}{r}\Delta B_z - ik\Delta B_\theta = \frac{4\pi}{c}\Delta J_r - \frac{i\omega}{c}\Delta E_r \tag{74}$$

where

$$\Delta E \equiv E_{\text{outside}} - E_{\text{inside}} \quad .$$

Using Eq. (70d), Eqs. (73d-f) become

$$\frac{i\ell}{r}\Delta E_z - ik\Delta E_\theta = 0 \tag{75a}$$

$$-ik\Delta E_r - \Delta E_z' = \frac{4\pi i\omega}{c^2} J_z^* \tag{75b}$$

$$\Delta E_\theta' + \frac{1}{r}\Delta E_\theta - \frac{i\ell}{r}\Delta E_r = -\frac{4\pi i\omega}{c^2} J_\theta^* \quad . \tag{75c}$$

Also Eqs. (73) and (74) are combined to give

$$\Delta J_r = i\omega\sigma^* - ikJ_z^* - \frac{i\ell}{r_0} J_\theta^* \quad . \tag{76}$$

Let

$$\sigma^* = \sigma_P^* + \sigma_B^* \tag{77a}$$

where σ_P^* is contributed by the plasma and σ_B^* is contributed by the beam. Since the radial current is only a small perturbation on the drift motion of the beam particles it is reasonable to suppose that, in the plasma frame of reference, the total surface current is due to the beam. Therefore it is assumed that

$$J_z^* = \sigma_B^* V_0 \tag{77b}$$

$$J_\theta^* = 0 \quad . \tag{77c}$$

Thus Eq. (76) becomes

$$\Delta J_r = \Delta J_r^P + \Delta J_r^B = i\omega\sigma_P^* + i\Omega\sigma_B^* \tag{78}$$

and the identification is made that

$$i\omega\sigma_P^* = \Delta J_r^P \tag{79a}$$

$$i\Omega\sigma_B^* = \Delta J_r^B \quad . \tag{79b}$$

Using Eqs. (57), (79), and (63)

$$4\pi\omega^2\sigma_P^* = \Delta[(P_1 + P_2)E_r + P_3E_z'] \tag{80a}$$

$$4\pi\omega\Omega\sigma_B^* = \Delta[B_1E_r + B_3E_z'] \tag{80b}$$

One boundary condition is obtained by combining Eqs. (80a,b) with Eq. (75b) and using $J_z^* = \sigma_B^* V_0$, to obtain

$$\Delta\left[\omega^2 - \sigma_1^2 - \frac{kV_0}{\Omega} B_1\right] ikE_r = \Delta\left[\sigma_2^2 + ik\frac{kV_0}{\Omega} B_3\right] E_z' \quad . \tag{81}$$

The other boundary conditions are given directly by Eqs. (73b), (73c), [Eq. (75a) is then automatically satisfied] and (75c).

Therefore, the regularity and boundary conditions for arbitrary normal mode are

a) $E_r, E_\theta, E_z, E_\theta', E_z'$ must be finite at $r = 0$ and $r = \infty$,

b) $E_\theta, E_z, E_\theta' + (i\ell/r_0)E_r$, and $[\omega^2 - \sigma_1^2 - (kV_0/\Omega)B_1]ikE_r - [\sigma_2^2 + ik(kV_0/\Omega)B_3]E_z'$ must be continuous at the boundary located at $r = r_0$. Hence, for $\ell \neq 0$ both the equations for the fields and the boundary conditions are coupled.

V DISPERSION RELATIONS FOR $\ell = 0$

The $\ell = 0$ mode is distinguished by the fact that both the differential equations for the field components and the boundary conditions can be uncoupled. Indeed Eqs. (70a-b) become, for $\ell = 0$,

$$E_z'' + \frac{2\zeta + 1}{r} E_z' + \beta^2 E_z = 0 \tag{82a}$$

$$E_\theta'' + \frac{1}{r} E_\theta' + \left[\eta^2 - \frac{1}{r^2}\right] E_\theta = 0 \tag{82b}$$

$$\eta^2 E_r = ik \left[1 - \frac{\sigma_2^2}{k^2 c^2}\right] E_z' \tag{82c}$$

where

$$\zeta = \frac{1}{2} \frac{(\sigma_2^2 - \sigma_4^2)(k^2c^2 - \sigma_2^2) - \sigma_6^2(\omega^2 - k^2c^2 - \sigma_1^2)}{(\omega^2 - k^2c^2 - \sigma_1^2)(k^2c^2 + \sigma_5^2 + \sigma_6^2) + (k^2c^2 - \sigma_2^2)^2} .$$

Equation (82c) has already been used to eliminate E_r from Eq. (82a). Using Eq. (82c) the boundary conditions are such that E_θ, E_z, E_θ', and WE_z' are continuous at $r = r_0$, where

$$W = \frac{\left(\omega^2 - \sigma_1^2 - \frac{kV_0}{\Omega} B_1\right)(k^2c^2 - \sigma_2^2) + (\omega^2 - k^2c^2 - \sigma_1^2)\left(\sigma_2^2 + ik \frac{kV_0}{\Omega} B_3\right)}{\omega^2 - k^2c^2 - \sigma_1^2} . \tag{83}$$

The solutions of Eqs. (82a,b) outside the beam depend upon the phases of β and η. If r_0 is the beam radius and R_0 the plasma radius then the notation will be used such that $\beta = \beta_s$ for $r < r_0$, β_L for $r_0 < r < R_0$ and similarly for η^2, σ_1^2, etc. If there is vacuum beyond R_0 then, for example, $\eta = \eta_r$ for $r > R_0$.

The solution of Eqs. (82a) and (82b) is straightforward. That is,

$$E_z(r) = r^{-\zeta} Z_\zeta(\beta r) \tag{84a}$$

where, $w = \beta r$, $Z_\zeta(w)$ satisfies the Bessel equation of order ζ,

$$Z''_\zeta(w) + \frac{1}{w} Z'_\zeta(w) + \left[1 - \frac{\zeta^2}{w^2}\right] Z_\zeta(w) = 0 \quad . \tag{84b}$$

Equation (82b) for $E_\theta(r)$ is also a Bessel equation, of order one.

The case of a plasma of infinite radius is discussed first. The solutions of Eqs. (82a,b) that satisfy the regularity conditions are

$$r < r_0: \quad E_z(r) = C_1 r^{-\zeta_s} J_{\zeta_s}(\beta_s r) \tag{85a}$$

$$E_\theta(r) = C_4 J_1(\eta_s r) \tag{85b}$$

$$r > r_0: \quad \text{Im } \beta_L > 0 \text{ or Im } \beta_L = 0, \text{ Re } \beta_L > 0, \text{ Re } \zeta_L \geq -\frac{1}{2}$$

$$E_z(r) = C_2 r^{-\zeta_L} H^{(1)}_{\zeta_L}(\beta_L r) \tag{85c}$$

$$\text{Im } \beta_L = 0 \text{ and Re } \zeta_L < -\frac{1}{2}$$

$$E_z(r) = 0 \tag{85d}$$

$$\beta_L = 0, \text{ Re } \zeta_L > 0 \text{ or Re } \zeta_L = 0, \text{ Im } \zeta_L \neq 0$$

$$E_z(r) = C_3 + C'_3 r^{-2\zeta_L} \tag{85e}$$

$$\beta_L = 0, \text{ Re } \zeta_L < 0 \text{ or } \zeta_L = 0$$

$$E_z(r) = C_3 \tag{85f}$$

$$\text{Im } \eta_L > 0 \text{ or Im } \eta_L = 0, \text{ Re } \eta_L > 0$$

$$E_\theta(r) = C_5 H^{(1)}_1(\eta_L r) \tag{85g}$$

$$\eta_L = 0$$

$$E_\theta(r) = C_6 \frac{1}{r} \quad . \tag{85h}$$

For Im $\beta_L < 0$ or Im $\beta_L = 0$, Re $\beta_L < 0$, and Re $\zeta_L \geq -(1/2)$ $H^{(2)}_{\zeta_L}(\beta_L r)$ must be substituted for $H^{(1)}_{\zeta_L}(\beta_L r)$ in Eq. (85c). For Im $\eta_L < 0$ or Im $\eta_L = 0$, Re $\eta_L < 0$ $H^{(1)}_1(\eta_L r)$ is changed to $H^{(2)}_2(\eta_L r)$ in Eq. (85g). These cases will henceforth be omitted since they lead to the same dispersion relations except for the interchange of Hankel functions.

The continuity conditions at $r = r_0$ give the following relations [where $J'(u) = dJ(u)/du$, etc.]:

Case Ia: Im $\beta_L > 0$ or Im $\beta_L = 0$, Re $\beta_L > 0$, and Re $\zeta_L \geq -\frac{1}{2}$

$$C_1 r_0^{-\zeta_s} J_{\zeta_s}(\beta_s r_0) = C_2 r_0^{-\zeta_L} H^{(1)}_{\zeta_L}(\beta_L r_0)$$

$$C_1 W_s r_0^{-\zeta_s - 1}[\beta_s r_0 J'_{\zeta_s}(\beta_s r_0) - \zeta_s J_{\zeta_s}(\beta_s r_0)] = C_2 W_L r_0^{-\zeta_L - 1}[\beta_L r_0 H^{(1)\prime}_{\zeta_L}(\beta_L r_0) - \zeta_L H^{(1)}_{\zeta_L}(\beta_L r_0)] \quad (86a)$$

Case Ib: Im $\beta_L = 0$ and Re $\zeta_L < -\frac{1}{2}$

$$C_1 r_0^{-\zeta_s} J_{\zeta_s}(\beta_s r_0) = 0$$

$$C_2 W_s r_0^{-\zeta_s - 1}[\beta_s r_0 J'_{\zeta_s}(\beta_s r_0) - \zeta_s J_{\zeta_s}(\beta_s r_0)] = 0 \quad (86b)$$

Case IIa: $\beta_L = 0$, Re $\zeta_L > 0$ or Re $\zeta_L = 0$, Im $\zeta_L \neq 0$

$$C_1 r_0^{-\zeta_s} J_{\zeta_s}(\beta_s r_0) = C_3 + C_3' r_0^{-2\zeta_L}$$

$$C_2 W_s r_0^{-\zeta_s - 1}[\beta_s r_0 J'_{\zeta_s}(\beta_s r_0) - \zeta_s J_{\zeta_s}(\beta_s r_0)] = -C_3' W_L 2\zeta_L r_0^{-2\zeta_L - 1} \quad (86c)$$

Case IIb: $\beta_L = 0$, Re $\zeta_L < 0$ or $\zeta_L = 0$

$$C_1 r_0^{-\zeta_s} J_{\zeta_s}(\beta_s r_0) = C_3$$

$$C_2 W_s r_0^{-\zeta_s - 1}[\beta_s r_0 J'_{\zeta_s}(\beta_s r_0) - \zeta_s J_{\zeta_s}(\beta_s r_0)] = 0 \quad (86d)$$

Case III: $\operatorname{Im} \eta_L > 0$ or $\operatorname{Im} \eta_L = 0$, $\operatorname{Re} \eta_L > 0$

$$C_4 J_1(\eta_s r_0) = C_5 H_1^{(1)}(\eta_L r_0)$$

$$C_4 \eta_s J_1'(\eta_s r_0) = C_5 \eta_L H_1^{(1)'}(\eta_L r_0) \qquad (86e)$$

Case IV: $\eta_L = 0$

$$C_4 J_1(\eta_s r_0) = C_6 \frac{1}{r_0}$$

$$C_4 \eta_s J_1'(\eta_s r_0) = -C_6 \frac{1}{r_0^2} \quad . \qquad (86f)$$

The dispersion relations that result from Eqs. (86a-f) are:

Case Ia: $\operatorname{Im} \beta_L > 0$ or $\operatorname{Im} \beta_L = 0$, $\operatorname{Re} \beta_L > 0$. $\operatorname{Re} \zeta_L \geq -\frac{1}{2}$, and

$$W_L \beta_L J_{\zeta_s}(\beta_s r_0) H^{(1)}_{\zeta_L+1}(\beta_L r_0) = W_s \beta_s J_{\zeta_s+1}(\beta_s r_0) H^{(1)}_{\zeta_L}(\beta_L r_0) \qquad (87a)$$

Case Ib: $\operatorname{Im} \beta_L = 0$, $\operatorname{Re} \zeta_L < -\frac{1}{2}$, and

$$J_{\zeta_s}(\beta_s r_0) = 0 = W_s \beta_s J_{\zeta_s+1}(\beta_s r_0) \qquad (87b)$$

Case IIa: $\beta_L = 0$, $\operatorname{Re} \zeta_L > 0$ or $\operatorname{Re} \zeta_L = 0$ and $\operatorname{Im} \zeta_L \neq 0$, and

$$2\zeta_L W_L J_{\zeta_s}(\beta_s r_0) = \beta_s r_0 W_s J_{\zeta_s+1}(\beta_s r_0) \qquad (87c)$$

Case IIb: $\beta_L = 0$, $\operatorname{Re} \zeta_L < 0$ or $\zeta_L = 0$, and

$$W_s \beta_s J_{\zeta_s+1}(\beta_s r_0) = 0 \qquad (87d)$$

Case III: $\operatorname{Im} \eta_L > 0$ or $\operatorname{Im} \eta_L = 0$, $\operatorname{Re} \eta_L > 0$, and

$$\eta_L J_1(\eta_s r_0) H_1^{(1)'}(\eta_L r_0) = \eta_s J_0(\eta_s r_0) H_1^{(1)}(\eta_L r_0) \qquad (87e)$$

Case IV: $\eta_L = 0$ and

$$\eta_s J_0(\eta_s r_0) = 0 \;, \tag{87f}$$

where the identities

$$zJ'_\nu(z) - \nu J_\nu(z) = - zJ_{\nu+1}(z)$$
$$zJ'_\nu(z) + \nu J_\nu(z) = zJ_{\nu-1}(z) \quad ,$$

which also hold for $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$, have been used.

Setting $\zeta = 0$ recovers the zero plasma temperature dispersion relations given in TI. Cases III and IV only depend upon the plasma temperature through η. Also it is easy to show that for $\eta_L = 0$ and hence, for Case IV, only damped or oscillatory solutions are allowed, at least for very low or very high plasma temperature. However, non-zero plasma temperature sufficiently complicates β_L so that no such conclusion can be reached about Cases Ib, IIa, or IIb. The reader is reminded that for *zero* plasma temperature $\beta_L = \eta_L$. Detailed analysis of these dispersion relations and those to follow will appear in the next report of this series.

For the case of a plasma bounded by a cylindrical conductor at $R_0 > r_0$ the solutions of Eqs. (82a,b) are (neglecting edge effects such as the plasma sheath)

$$r < r_0: \quad E_z(r) = C_1 r^{-\zeta_s} J_{\zeta_s}(\beta_s r) \tag{88a}$$

$$E_\theta(r) = C_4 J_1(\eta_s r) \tag{88b}$$

$r_0 < r < r_0$: $\beta_L \neq 0$

$$E_z(r) = r^{-\zeta_L}[C_2 H_{\zeta_L}^{(1)}(\beta_L r) + C_2' H_{\zeta_L}^{(2)}(\beta_L r)] \tag{88c}$$

$\beta_L = 0, \; \zeta_L \neq 0$

$$E_z(r) = C_3 + C_3' r^{-2\zeta_L} \tag{88d}$$

$\beta_L = 0, \; \zeta_L = 0$

$$E_z(r) = C_3 + C_3'' \; ln \; r \tag{88e}$$

$\eta_L \neq 0$

$$E_\theta(r) = C_5 H_1^{(1)}(\eta_L r) + C_5' H_1^{(2)}(\eta_L r) \tag{88f}$$

$\eta_L = 0$

$$E_\theta(r) = C_6 \frac{1}{r} + C_6' r \quad . \tag{88g}$$

The boundary conditions at the wall (assuming a perfect conductor) are such that the tangential component of $\vec{E}$ and the normal component of $\vec{B}$ are continuous: both are satisfied for arbitrary normal mode if $E_\theta(R_0) = 0 = E_z(R_0)$.

The resulting dispersion relations are

$\beta_L \neq 0$ and

$$\beta_L W_L J_{\zeta_s}(\beta_s r_0)[H_{\zeta_L}^{(1)}(\beta_L R_0) H_{\zeta_L+1}^{(2)}(\beta_L r_0) - H_{\zeta_L+1}^{(1)}(\beta_L r_0) H_{\zeta_L}^{(2)}(\beta_L R_0)]$$

$$= \beta_s W_s J_{\zeta_s+1}(\beta_s r_0)[H_{\zeta_L}^{(1)}(\beta_L R_0) H_{\zeta_L}^{(2)}(\beta_L r_0) - H_{\zeta_L}^{(1)}(\beta_L r_0) H_{\zeta_L}^{(2)}(\beta_L R_0)] \tag{89a}$$

$\beta_L = 0,\ \zeta_L \neq 0$ and

$$2\zeta_L W_L J_{\zeta_s}(\beta_s r_0) = W_s \beta_s r_0 J_{\zeta_s+1}(\beta_s r_0) \left(1 - \frac{r_0}{R_0}\right)^{2\zeta_L} \tag{89b}$$

$\beta_L = 0,\ \zeta_L = 0$ and

$$W_L J_{\zeta_s}(\beta_s r_0) = W_s \beta_s r_0 \ ln \frac{R_0}{r_0} J_{\zeta_s+1}(\beta_s r_0) \tag{89c}$$

$\eta_L \neq 0$ and

$$\eta_L J_1(\eta_s r_0)[H_0^{(1)}(\eta_L r_0) H_1^{(2)}(\eta_L R_0) - H_1^{(1)}(\eta_L R_0) H_0^{(2)}(\eta_L r_0)]$$

$$= \eta_s J_0(\eta_s r_0)[H_1^{(1)}(\eta_L r_0) H_1^{(2)}(\eta_L R_0) - H_1^{(1)}(\eta_L R_0) H_1^{(2)}(\eta_L r_0)] \tag{89d}$$

$\eta_L \;=\; 0$ and

$$\eta_s(R_0^2 - r_0^2)J_0(\eta_s r_0) \;=\; -2r_0J_1(\eta_s r_0) \quad . \tag{89e}$$

The final case to be treated is that of a plasma extending out to R_0 beyond which is vacuum. The fields are given by Eqs. (88a-g) for $r < R_0$ and, for $r > R_0$,

Im $\eta_\nu > 0$ or Im $\eta_\nu \;=\; 0$, Re $\eta_\nu > 0$

$$E_z(r) \;=\; C_7H_0^{(1)}(\eta_\nu r) \tag{90a}$$

$$E_\theta(r) \quad C_8H_1^{(1)}(\eta_\nu r) \tag{90b}$$

$\eta_\nu \;=\; 0$

$$E_z(r) \;=\; C_{11} \tag{90c}$$

$$E_\theta(r) \;=\; C_{12}\frac{1}{r} \tag{90d}$$

where

$$C^2\eta_\nu^2 \;=\; \omega^2 - k^2c^2 \quad .$$

Since it has been assumed that in its own rest frame the plasma produces no surface current then at the plasma-vacuum interface,

$$\frac{\omega^2 - \sigma_1^2 - \sigma_2^2}{\omega^2 - k^2c^2 - \sigma_1^2}E_z' \quad ,$$

is continuous, instead of WE_z'.

Thus the dispersion relations are

$\beta_L \;\neq\; 0$

Im $\eta_\nu > 0$ or

Im $\eta_\nu \;=\; 0$,

Re $\eta_\nu > 0$,

and

$$\beta_L W_L J_{\zeta_s}(\beta_s r_0)\left\{\beta_L \frac{\omega^2-\sigma_{1L}^2-\sigma_{2L}^2}{\omega^2-k^2c^2-\sigma_{1L}^2} H_0^{(1)}(\eta_\nu R_0)[H_{\zeta_L+1}^{(1)}(\beta_L r_0)H_{\zeta_L+1}^{(2)}(\beta_L R_0)-H_{\zeta_L+1}^{(1)}(\beta_L R_0)H_{\zeta_L+1}^{(2)}(\beta_L r_0)]\right.$$

$$\left.+\eta_\nu \frac{\omega^2}{\omega^2-k^2c^2} H_1^{(1)}(\eta_\nu R_0)[H_{\zeta_L}^{(1)}(\beta_L R_0)H_{\zeta_L+1}^{(2)}(\beta_L r_0)-H_{\zeta_L}^{(2)}(\beta_L R_0)H_{\zeta_L+1}^{(1)}(\beta_L r_0)]\right\}$$

$$=\beta_s W_s J_{\zeta_s+1}(\beta_s r_0)\left\{\beta_L \frac{\omega^2-\sigma_{1L}^2-\sigma_{2L}^2}{\omega^2-k^2c^2-\sigma_{1L}^2} H_0^{(1)}(\eta_\nu R_0)[H_{\zeta_L}^{(1)}(\beta_L r_0)H_{\zeta_L+1}^{(2)}(\beta_L R_0)-H_{\zeta_L+1}^{(1)}(\beta_L R_0)H_{\zeta_L}^{(2)}(\beta_L r_0)]\right.$$

$$\left.+\eta_\nu \frac{\omega^2}{\omega^2-k^2c^2} H_1^{(1)}(\eta_\nu R_0)[H_{\zeta_L}^{(1)}(\beta_L R_0)H_{\zeta_L}^{(2)}(\beta_L R_0)-H_{\zeta_L}^{(1)}(\beta_L r_0)H_{\zeta_L}^{(2)}(\beta_L R_0)]\right\} \quad (91a)$$

$\beta_L \neq 0$, $\eta_\nu = 0$, and

$$\beta_L W_L J_{\zeta_s}(\beta_s r_0)[H_{\zeta_L+1}^{(1)}(\beta_L R_0)H_{\zeta_L+1}^{(2)}(\beta_L r_0)-H_{\zeta_L+1}^{(1)}(\beta_L r_0)H_{\zeta_L+1}^{(2)}(\beta_L R_0)]$$

$$=\beta_s W_s J_{\zeta_s+1}(\beta_s r_0)[H_{\zeta_L+1}^{(1)}(\beta_L R_0)H_{\zeta_L}^{(2)}(\beta_L r_0)-H_{\zeta_L}^{(1)}(\beta_L r_0)H_{\zeta_L+1}^{(2)}(\beta_L R_0)] \quad (91b)$$

$\beta_L = 0$, $\zeta_L \neq 0$, and Im $\eta_\nu >$ or Im $\eta_\nu = 0$, Re $\eta_\nu > 0$ and

$$2\zeta_L W_L \eta_\nu \frac{\omega^2}{\omega^2-k^2c^2} J_{\zeta_s}(\beta_s r_0)H_1^{(1)}(\eta_\nu R_0)$$

$$=\beta_s W_s J_{\zeta_s+1}(\beta_s r_0)\left\{2\zeta_L \frac{\omega^2-\sigma_{1L}^2-\sigma_{2L}^2}{\omega^2-k^2c^2-\sigma_{1L}^2}\left(\frac{r_0}{R_0}\right)^{2\zeta_L+1} H_0^{(1)}(\eta_\nu R_0)\right.$$

$$\left.+\eta_\nu R_0 \frac{\omega^2}{\omega^2-k^2c^2} H_1^{(1)}(\eta_\nu R_0)\left[1-\left(\frac{r_0}{R_0}\right)^{2\zeta_L}\right]\right\} \quad (91c)$$

$\beta_L = 0$, $\zeta_L = 0$, and Im $\eta_\nu > 0$ or Im $\eta_\nu = 0$, Re $\eta_\nu > 0$ and

$$W_L \eta_\nu \frac{\omega^2}{\omega^2-k^2c^2} J_{\zeta_s}(\beta_s r_0)H_1^{(1)}(\eta_\nu R_0)$$

$$=W_s \beta_s J_{\zeta_s+1}(\beta_s r_0)\left[\frac{\omega^2-\sigma_{1L}^2-\sigma_{2L}^2}{\omega^2-k^2c^2-\sigma_{1L}^2}\frac{r_0}{R_0}H_0^{(1)}(\eta_\nu R_0)+\eta_\nu r_0 \frac{\omega^2}{\omega^2-k^2c^2} \ln\frac{R_0}{r_0} H_1^{(1)}(\eta_\nu R_0)\right] \quad (91d)$$

$\beta_L = 0, \eta_\nu = 0,$ and

$$\beta_s W_s \left(\frac{1}{R_0}\right)^{2\zeta_L+1} \frac{\omega^2 - \sigma_{1L}^2 - \sigma_{2L}^2}{\omega^2 - k^2c^2 - \sigma_{1L}^2} J_{\zeta_s+1}(\beta_s r_0) = 0 \tag{91e}$$

$\eta_L \neq 0$ and Im $\eta_\nu > 0$ or Im $\eta_\nu = 0$, Re $\eta_\nu > 0$ and

$$\begin{aligned}
&\eta_L J_1(\eta_s r_0)\{\eta_\nu H_0^{(1)}(\eta_\nu R_0)[H_0^{(1)}(\eta_L r_0)H_1^{(2)}(\eta_L R_0) \\
&- H_0^{(2)}(\eta_L r_0)H_1^{(1)}(\eta_L R_0)] \\
&+ \eta_L H_1^{(1)}(\eta_\nu R_0)[H_0^{(1)}(\eta_L R_0)H_0^{(2)}(\eta_L r_0) - H_0^{(1)}(\eta_L r_0)H_0^{(2)}(\eta_L R_0)]\} \\
&= \eta_s J_0(\eta_s r_0)\{\eta_\nu H_0^{(1)}(\eta_\nu R_0)[H_1^{(1)}(\eta_L r_0)H_1^{(2)}(\eta_L R_0) \\
&- H_1^{(1)}(\eta_L R_0)H_1^{(2)}(\eta_L r_0)] \\
&+ \eta_L H_1^{(1)}(\eta_\nu R_0)[H_0^{(1)}(\eta_L R_0)H_1^{(2)}(\eta_L r_0) - H_1^{(1)}(\eta_L r_0)H_0^{(2)}(\eta_L R_0)]\}
\end{aligned} \tag{91f}$$

$\eta_L = 0$ and Im $\eta_\nu > 0$ or Im $\eta_\nu = 0$, Re $\eta_\nu > 0$ and

$$\begin{aligned}
2\eta_\nu r_0 J_1(\eta_s r_0)H_0^{(1)}(\eta_\nu R_0) = \eta_s J_0(\eta_s r_0)[&2R_0 H_1^{(1)}(\eta_\nu R_0) \\
&- \eta_\nu (R_0^2 - r_0^2)H_0^{(1)}(\eta_\nu R_0)]
\end{aligned} \tag{91g}$$

$\eta_L \neq 0, \eta_\nu \equiv 0,$ and

$$\begin{aligned}
&\eta_L J_1(\eta_s r_0)[H_0^{(1)}(\eta_L R_0)H_0^{(2)}(\eta_L r_0) - H_0^{(1)}(\eta_L r_0)H_0^{(2)}(\eta_L R_0)] \\
&= \eta_s J_0(\eta_s r_0)[H_0^{(1)}(\eta_L R_0)H_1^{(2)}(\eta_L r_0) - H_1^{(1)}(\eta_L r_0)H_0^{(2)}(\eta_L R_0)]
\end{aligned} \tag{91h}$$

$$\eta_L = 0, \quad \eta_\nu = 0, \text{ and}$$

$$\eta_s R_0 J_0(\eta_s r_0) = \left(\frac{R_0}{r_0} - 1 \right) J_1(\eta_s r_0) \quad . \tag{91i}$$

It should be noted that the cases with $\eta_\nu = 0 = \omega^2 - k^2c^2$, Eqs. (91e), (91h), and (91i), have only oscillatory solutions.

VI DISPERSION RELATIONS FOR ZERO PLASMA TEMPERATURE AND $\ell \neq 0$

When $T_e = 0 = T_i$ great simplification occurs for $\ell \neq 0$ since then $\zeta_1 = 0 = \zeta_2$ and Eqs. (70a-c) become

$$E_z'' + \frac{1}{r} E_z' + \left[\beta^2 - \frac{\ell^2}{r^2}\right] E_z = 0 \tag{92a}$$

$$E_r'' + \frac{3}{r} E_r' + \left[\eta^2 - \frac{\ell^2 - 1}{r^2}\right] E_r = \frac{\alpha}{\eta^2}\left[(\beta^2 - \eta^2) E_z' + 2\frac{\beta^2}{r} E_z\right] \tag{92b}$$

$$-\frac{i\ell}{r} E_\theta' = E_r' + \frac{1}{r} E_r + \alpha \frac{\beta^2}{\eta^2} E_z \tag{92c}$$

where

$$\alpha = \frac{i}{kc^2}(k^2c^2 - \sigma_2^2) \quad .$$

The solution of Eqs. (92a-c) is straightforward. Only the case of an infinite plasma will be considered here. The fields are given by

$$r < r_0: \qquad E_z(r) = C_1 J_\ell(\beta_s r) \tag{93a}$$

$$E_r(r) = C_4 \frac{1}{r} J_\ell(\eta_s r) + C_1 \frac{\alpha_s}{\eta_s^2} \frac{1}{r}\left[\frac{\beta_s^2(1+\ell) - \ell\eta_s^2}{\eta_s^2 - \beta_s^2} J_\ell(\beta_s r) + \beta_s r J_{\ell-1}(\beta_s r)\right] \tag{93b}$$

$$-i\ell E_\theta(r) = C_4\eta_s J'_\ell(\eta_s r) + C_1\frac{\alpha_s\beta_s}{\eta_s^2}\left[(\beta_s r-\ell)J_\ell(\beta_s r) - \frac{1}{2}\frac{\beta_s^2(1+\ell)-\ell\eta_s^2}{\eta_s^2-\beta_s^2}J_{\ell+1}(\beta_s r)\right.$$

$$\left. + \frac{(\eta_s^2-\beta_s^2)(\beta_s r+1) + \frac{1}{2}\beta_s^2(1+\ell) - \frac{1}{2}\ell\eta_s^2}{\eta_s^2-\beta_s^2} J_{\ell-1}(\beta_s r)\right] \quad (93c)$$

$r > r_0$: (recall that, for zero plasma temperature, $\beta_L^2 = \eta_L^2$) .

$\text{Im}\ \beta_L > 0$ or $\text{Im}\ \beta_L = 0$, $\text{Re}\ \beta_L > 0$

$$E_z(r) = C_2H_\ell^{(1)}(\beta_L r) \quad (93d)$$

$$E_r(r) = C_5\frac{1}{r}H_\ell^{(1)}(\beta_L r) - c_2\frac{\alpha_L}{\beta_L}H_\ell^{(1)}(\beta_L r) \quad (93e)$$

$$-i\ell E_\theta(r) = C_5\beta_L H_\ell^{(1)'}(\beta_L r) + c_2\frac{\alpha_L}{\beta_L}\left[2\beta_L r - \frac{\ell^2}{\beta_L r}\right]H_\ell^{(1)}(\beta_L r) \quad (93f)$$

$\beta_L = 0$

$$E_z(r) = C_3 r^{-\ell} \quad (93g)$$

$$E_r(r) = C_6 r^{-\ell-1} \quad (93h)$$

$$i\ell E_\theta(r) = -C_6 m r^{-\ell-1} + C_3\alpha_L r^{-\ell-1} \quad . \quad (93i)$$

The dispersion relations are obtained by using the general boundary conditions given in Section IV. Due to the very complex structure of these dispersion relations, they are best left in determinant form. Since $\beta_L = 0$ implies only oscillatory or damped solutions, this case will be neglected. Hence for $\text{Im}\ \beta_L > 0$ or $\text{Im}\ \beta_L = 0$, $\text{Re}\ \beta_L > 0$ the dispersion relation is

$$J_\ell(\beta_s r_0)\ \text{Det}\ (A_{ik}) + H_\ell^{(1)}(\beta_L r_0)\ \text{Det}\ (B_{ik}) = 0 \quad (94)$$

where

$$A_{11} = \frac{\alpha_L}{\beta_L}\,\Omega(\omega^2 - P_{1L})H_\ell^{(1)'}(\beta_L r_0)$$

$$A_{12} = B_{12} = (\omega^2\Omega - P_{1s}\Omega - \omega B_{1s})\frac{1}{r_0}\,J_\ell(\eta_s r_0)$$

$$A_{13} = B_{13} = -\Omega(\omega^2 - P_{1L})\,\frac{1}{r_0}\,H_\ell^{(1)}(\beta_L r_0)$$

$$A_{21} = \alpha_L r_0\left(\frac{\ell^2}{\beta_L^2 r_0^2} - 2\right) H_\ell^{(1)}(\beta_L r_0)$$

$$A_{22} = B_{22} = \eta_s J_\ell'(\eta_s r_0)$$

$$A_{23} = B_{23} = -\beta_L H_\ell^{(1)'}(\beta_L r_0)$$

$$A_{31} = \frac{\alpha_L}{\beta_L^2 r_0^2}\Big\{\beta_L r_0 \ell^2 H_\ell^{(1)'}(\beta_L r_0) - [2(\ell+1)\beta_L^2 r_0^2 - \ell^2]H_\ell^{(1)}(\beta_L r_0)$$

$$-\beta_L r_0(\ell - 2\beta_L^2 r_0^2)H_{\ell+1}^{(1)}(\beta_L r_0)\Big\}$$

$$A_{32} = B_{32} = \eta_s^2 J_\ell''(\eta_s r_0) + \frac{\ell^2}{r_0^2}\,J_\ell(\eta_s r_0)$$

$$A_{33} = B_{33} = -\beta_L^2 H_\ell^{(1)''}(\beta_L r_0) + \frac{\ell^2}{r_0^2}\,H_\ell^{(1)}(\beta_L r_0)$$

$$B_{11} = (\omega^2\Omega - P_{1s}\Omega - \omega B_{1s})\frac{\alpha_s}{\eta_s^2 r_0}\left[\frac{\beta_s^2(1 + \ell) - \ell\eta_s^2}{\eta_s^2 - \beta_s^2}\,J_\ell(\beta_s r_0)\right.$$

$$\left. + \beta_s r_0 J_{\ell-1}(\beta_s r_0) - \frac{\omega\beta_s\eta_s^2 r_0 B_{3s}}{\omega^2\Omega - P_{1s}\Omega - \omega B_{1s}}\,\frac{1}{\alpha_s}\,J_\ell'(\beta_s r_0)\right]$$

$$B_{21} = \frac{\alpha_s \beta_s}{\eta_s^2} \left[(\beta_s r_0 - \ell) J_\ell(\beta_s r_0) - \frac{1}{2} \frac{\beta_s^2 (1 + \ell) - \ell \eta_s^2}{\eta_s^2 - \beta_s^2} J_{\ell+1}(\beta_s r_0) \right.$$

$$\left. + \frac{(\eta_s^2 - \beta_s^2)(\beta_s r_0 + 1) + \frac{1}{2} \beta_s^2 (1 + \ell) - \frac{1}{2} \ell \eta_s^2}{\eta_s^2 - \beta_s^2} J_{\ell-1}(\beta_s r_0) \right]$$

and

$$B_{31} = \frac{\alpha_s}{\eta_s^2} \left[\beta_s^2 (1 + \ell - \beta_s r_0) J_\ell(\beta_s r_0) + \frac{1}{2} \beta_s^2 (\ell - 2\beta_s r_0) J_{\ell+1}(\beta_s r_0) \right.$$

$$+ \frac{1}{2} \beta_s^2 (\ell + 2) J_{\ell-1}(\beta_s r_0) + \beta_s^2 J'_{\ell-1}(\beta_s r_0) + \ell^2 \frac{\beta_s}{r_0} J_{\ell-1}(\beta_s r_0)$$

$$\left. + \frac{\beta_s^2 (1 + \ell) - \ell \eta_s^2}{\eta_s^2 - \beta_s^2} \beta_s^2 J''_\ell(\beta_s r_0) + \frac{\beta_s^2 (1 + \ell) - \ell \eta_s^2}{\eta_s^2 - \beta_s^2} \frac{\ell^2}{r_0^2} J_\ell(\beta_s r_0) \right] .$$

It is quite apparent that the analysis of Eq. (94) presents a very formidable task. Therefore, it is of some interest to consider some special modes for $\ell \neq 0$ that result in simpler dispersion relations. This is taken up in the next section.

VII SPECIAL MODES FOR NON-ZERO PLASMA TEMPERATURE AND $\ell \neq 0$

Since the complete investigation of the $\ell \neq 0$ and non-zero plasma temperature cases necessitates the solution of a formidable sixth-order differential equation for $E_z(r)$, Eq. (72), it is of some interest to look at some special modes in which the analysis can be considerably simplified. The pure longitudinal and pure transverse modes are treated in this section.

A. PURE LONGITUDINAL MODES

Here $E_\theta = 0 = E_r$ and Eqs. (70a-c) become

$$ik[k^2c^2 - \sigma_2^2] \frac{i\ell}{r} E_z = 0 \tag{95a}$$

$$ik[k^2c^2 - \sigma_2^2]E_z' = 0 \tag{95b}$$

$$E_z'' + \frac{2a + 1}{r} E_z' + \left[b^2 - \frac{\ell^2}{r^2}\right]E_z = 0 \tag{95c}$$

where

$$a = -\frac{1}{2} \frac{\sigma_6^2}{k^2c^2 + \sigma_5^2 + \sigma_6^2}$$

$$b^2 = k^2 \frac{\omega^2 - \sigma_3^2}{k^2c^2 + \sigma_5^2 + \sigma_6^2} .$$

Hence Eqs. (95a,b) show that, for any ℓ, pure longitudinal waves are possible if

$$k^2c^2 = \sigma_{2s}^2 \tag{96a}$$

or

$$k^2c^2 = \sigma_{2L}^2 \quad . \tag{96b}$$

The fields for an infinite plasma, if Eq. (96a) holds, are

$$r < r_0: \qquad E_z(r) = C_1 r^{-a_S} J_{A_S}(b_S r) \tag{97a}$$

$$r > r_0: \qquad E_z(r) = 0 \tag{97b}$$

or, if Eq. (96b) holds,

$$r < r_0: \qquad E_z(r) = 0 \tag{97c}$$

$r > r_0$: Im $b_L > 0$ or Im $b_L = 0$, Re $b_L > 0$

$$E_z(r) = C_2 r^{-a_L} H_{A_L}^{(1)}(b_L r) \tag{97d}$$

$b_L = 0$

$$E_z(r) = C_3 r^{-a_L - A_L} \tag{97e}$$

where

$$A = [a^2 + \ell^2]^{1/2} \quad .$$

For a plasma bounded by a perfect conductor of radius R_0, the only changes in the solutions for the fields are for $r_0 < r < R_0$ when Eq. (96b) holds. In this case change Eq. (97d) to

$$E_z(r) = r^{-a_L}[C_2 H_{A_L}^{(1)}(b_L r) + C' H_{A_L}^{(2)}(b_L r)] \tag{98a}$$

and change Eq. (97e) to

$$E_z(r) = r^{-a_L}[C_3 r^{-A_L} + C_3' r^{A_L}] \quad . \tag{98b}$$

For a plasma extending out to R_0 beyond which is vacuum the fields are given by Eqs. (97a-c), Eqs. (98a,b), and for $r > R_0$,

$$E_z(r) = 0 \quad . \tag{99}$$

The general boundary conditions for arbitrary ℓ are given in Section IV. They reduce in this case to the continuity of E_z and $[\sigma_2^2 + ik(kV_0/\Omega)B_3]E_z'$ at the beam radius r_0, to the condition $E_z(R_0) = 0$ at the surface of the conductor, and to the continuity of E_z and E_z' [using Eq. (95b)] at the plasma-vacuum interface.

The dispersion relations for an infinite plasma are

Case I:

$$k^2c^2 - \sigma_{2S}^2 = 0 \tag{100a}$$

$$J_{A_S}(b_S r_0) = 0 \tag{100b}$$

$$\left(\sigma_{2S}^2 + ik\frac{kV_0}{\Omega} B_{3S}\right) [(A_S - a_S)J_{A_S}(b_S r_0) - b_S r_0 J_{A_S+1}(b_S r_0)] = 0 \tag{100c}$$

and

Case II: $\quad \mathrm{Im}\ b_L > 0 \quad$ or $\quad \mathrm{Im}\ b_L = 0,\ \mathrm{Re}\ b_L > 0$

$$k^2c^2 - \sigma_{2L}^2 = 0 \tag{101a}$$

$$H_{A_L}^{(1)}(b_L r_0) = 0 \tag{101b}$$

$$\sigma_{2L}^2[(A_L - a_L)H_{A_L}^{(1)}(b_L r_0) - b_L r_0 H_{A_L+1}^{(1)}(b_L r_0)] = 0 \quad . \tag{101c}$$

When $b_L = 0$, the continuity of E_z requires $C_3 = 0$. Since combining Eqs. (101a-c) implies a contradiction, namely $b_L = 0$, then only Case I survives. Therefore, combining Eqs. (100a-c) gives the dispersion relation for pure longitudinal waves in an infinite plasma:

$$\left(1 + i\frac{V_0}{\Omega C^2} B_3\right)b_S = 0 \quad . \tag{102}$$

The dispersion relations for a conductor at R_0 reduce to the same result as for an infinite plasma, Eq. (102), since the same contradiction is easily found to hold as above (*i.e.*, $\mathrm{Im}\ b_L > 0$ or $\mathrm{Im}\ b_L = 0$, $\mathrm{Re}\ b_L > 0$ *and* $b_L = 0$) when Eq. (96b) holds. A similar result is also found for the beam-plasma-vacuum system.

Hence, the only possible pure longitudinal mode must satisfy the dispersion relation, Eq. (102). This mode is characterized by having $E_z(r) = 0$ everywhere outside the beam.

The solution to Eq. (102) corresponding to $1 + i(V_0/\Omega C^2)B_3 = 0$ is equivalent to

$$1 - \frac{kV_0}{\Omega}\,\frac{\omega_\perp^2}{k^2c^2}\left[1 + \frac{\omega + i\nu_B}{kU_B}\,Z_B\right] \quad . \tag{103}$$

For zero beam temperature Eq. (103) reduces to [using $\omega_\perp^2 = 2(C^2/V_0^2)\omega_\beta^2$]

$$1 + \frac{2\omega_\beta^2}{\Omega(\Omega + i\nu_B)} \simeq 0 \tag{104a}$$

or

$$\Omega(\Omega + i\nu_B) + 2\omega_\beta^2 \simeq 0 \quad . \tag{104b}$$

The high-frequency criterion, $|\Omega|^2 >> \omega_\beta^2$, must be imposed. Therefore, Eq. (104b) can be satisfied only if $\nu_B >> \omega_\beta$. In this case, a damped solution results:

$$\Omega \simeq -i\nu_B - i2\,\frac{\omega_\beta^2}{\nu_B} \quad . \tag{104c}$$

For high beam temperature Eq. (103) becomes (to first order in $1/U_B$)

$$1 - \frac{kV_0}{\Omega}\,\frac{\omega_\perp^2}{k^2c^2}\left[1 + i\sqrt{\frac{\pi}{2}}\,\frac{\omega + i\nu_B}{kU_B}\right] = 0 \quad . \tag{105a}$$

For $V_0 \simeq c$, the solution of Eq. (105a) is

$$\omega = \frac{kV_0 - \dfrac{\omega_\perp^2}{kc}\sqrt{\dfrac{\pi}{2}}\,\dfrac{\nu_B}{kU_B}}{1 - i\sqrt{\dfrac{\pi}{2}}\,\dfrac{\omega_\perp^2}{k^2U_B^2}} \quad . \tag{105b}$$

Since for high temperatures, $\omega_\perp^2/k^2U_B^2 << 1$, then it is easily seen that Eqs. (105b) does not satisfy $|\Omega|^2 >> \omega_\beta^2 \approx (1/2)\omega_\perp^2$. This means that for extremely low or extremely high beam temperatures, Eq. (103) possesses no consistent, unstable solutions.

To obtain the other dispersion equation of Eq. (102) we let $b_\mathcal{S} \to 0$. From Eq. (97a,b) this implies that the field vectors are zero everywhere unless $A_\mathcal{S} \to 0$, which in turn, implies that $\ell = 0$ and a zero-temperature plasma. For finite beam radius, r_0, and when $a_\mathcal{S} = A_\mathcal{S} = b_\mathcal{S} \to 0$, Eqs. (97a) and (97b) yield

$$E_z(r) = c_1, \qquad r < r_0$$

$$E_z(r) = 0\,, \qquad r > r_0 \quad .$$

By Eq. (73c) $\Delta E_z = 0$, so that $c_1 = 0$, and the field vectors are zero everywhere again. Thus, the above set of values yields no physically interesting situation.

Another way of seeing this is to note that Eq. (100b) cannot be satisfied as $a_\mathcal{S} = A_\mathcal{S} = b_\mathcal{S} \to 0$, for finite r_0, for then $J_{A_\mathcal{S}}(b_\mathcal{S} r_0) = J_0(0) = 1$ and not zero as required by Eq. (100b). However, as $r_0 \to \infty$ and when $a_S = A_S = b_S = x_\mathcal{S}/r_0 \to 0$ so that $b_\mathcal{S} r_0 = x_\mathcal{S}$, where $J_0(x_\mathcal{S}) = 0$, Eq. (100b) can be satisfied. In this case, by virtue of Eq. (47a) of TI, $r_0 \to \infty$, $b_S = x_\mathcal{S}/r_0 \to 0$, yields

$$1 \approx \frac{\omega_p^2}{\omega^2} - \frac{\omega_\parallel^2}{\omega^2} - \frac{\omega(\omega + i\nu_B)}{k^2U_B^2}\left[1 + \frac{\Omega + i\nu_B}{kU_B}\, Z_B\right] \quad , \tag{106}$$

the dispersion equation for an infinite beam in an infinite cold plasma.

The dispersion relation, Eq. (106), has been exhaustively studied for $\nu_B = 0$ by H. Singhaus.[5] His main result is that, if the beam temperature is high enough [*i.e.*, $\tau \equiv (\omega_B/\omega_\perp)(U_B/V_0) >> 1$], then the only instability possible is characterized by $|\omega| \approx \omega_e \approx kV_0$ and a growth rate $\mathrm{Im}\,\omega \simeq (1/4\tau^2)\omega_e$. However, the instability is quenched if $\nu_e \gtrsim (\sqrt{\pi/2e}) \bullet (\omega_e/\tau^2)$, where ν_e is the plasma collision frequency.

B. PURE TRANSVERSE MODES

From Eq. (70c) it is immediately evident that $E_\theta = 0 = E_z$ and $E_r \neq 0$ is impossible. However if $E_r = 0 = E_z$ then Eqs. (70a-c) become

$$[k^2c^2 - \sigma_2^2]\,\frac{i\ell}{r}E_\theta = 0 \tag{107a}$$

$$E_\theta'' + \frac{1}{r}E_\theta' + \left[\eta^2 - \frac{1}{r^2}\right]E_\theta = 0 \tag{107b}$$

$$\frac{i\ell}{r}\left[E_\theta' + \frac{1}{r}E_\theta\right] = 0 \quad . \tag{107c}$$

Combining Eqs. (107b) and (107c)

$$i\ell\eta^2 E_\theta = 0 \quad . \tag{107d}$$

For $\ell = 0$ the dispersion relations are found by solving Eq. (107b) and this has been given in Section V. For $\ell \neq 0$ a non-trivial solution is obtained only if Eq. (107d) is satisfied by having $\eta^2 = 0$ and Eq. (107a) is satisfied by setting $k^2c^2 - \sigma_2^2$. This is equivalent to either

$$\omega^2 - \sigma_{1s}^2 - \sigma_{2s}^2 = 0 \tag{108a}$$

or

$$\omega^2 - \sigma_{1L}^2 - \sigma_{2L}^2 = 0 \quad . \tag{108b}$$

When Eq. (108a) holds, $E_\theta(r) = C_4 r$ for $r < r_0$ and $E_\theta(r) = 0$ everywhere else. When Eq. (108b) holds, $E_\theta(r) = C_5 r + C_5'/r$ for $r_0 < r < R_0$ and $E_\theta(r) = 0$ everywhere else. Continuity of E_θ and E_θ' then implies that $E_\theta(r)$ must be zero everywhere for both cases.

Hence pure transverse waves with just one cylindrical component, either E_r or E_θ, are impossible for $\ell \neq 0$.

If both $E_r \neq 0$ and $E_\theta \neq 0$ when $E_z = 0$ then Eqs. (70a-c) become

$$[k^2c^2 - \sigma_2^2]\left(E_r' + \frac{i\ell}{r}E_\theta\right) + [k^2c^2 - \sigma_4^2]\frac{1}{r}E_r = 0 \tag{109a}$$

$$E_\theta'' + \frac{1}{r} E_\theta' + \left[\eta^2 - \frac{1}{r^2}\right] E_\theta - \frac{i\ell}{r}\left(E_r' - \frac{1}{r} E_r\right) = 0 \qquad (109b)$$

$$\left[\eta^2 - \frac{\ell^2}{r^2}\right] E_r - \frac{i\ell}{r}\left(E_\theta' + \frac{1}{r} E_\theta\right) = 0 \quad . \qquad (109c)$$

Combining Eqs. (109b) and (109c)

$$-\frac{i\ell}{r} E_\theta = E_r' + \frac{1}{r} E_r \quad . \qquad (109d)$$

Thus Eqs. (109d) and (109a) give

$$(\sigma_2^2 - \sigma_4^2)\frac{1}{r} E_r = 0 \quad . \qquad (109e)$$

Now $\sigma_2^2 - \sigma_4^2 = -k^2 P_5$ so $E_r \neq 0$ only if

$$P_{5S} = 0 \qquad (110a)$$

or

$$P_{5L} = 0 \quad , \qquad (110b)$$

where it is assumed that the presence of the beam causes some discontinuity, however small, in the plasma parameters.

When $\sigma_2^2 = \sigma_4^2$ is inserted into Eqs. (109a-c), an uncoupled equation for $E_r(r)$ is obtained:

$$E_r'' + \frac{3}{r} E_r' + \left[\eta^2 - \frac{\ell^2 - 1}{r^2}\right] E_r = 0 \quad . \qquad (111)$$

Attention is again focussed upon $\ell \neq 0$ since the $\ell = 0$ case has been exhaustively treated in Section V. The fields, for an infinite plasma, when $P_{5S} = 0$, are

$$r < r_0: \qquad E_r(r) = C_4 \frac{1}{r} J_\ell(\eta_S r) \qquad (112a)$$

$$-i\ell E_\theta(r) = C_4 \eta_S J_\ell(\eta_S r) \qquad (112b)$$

$$r > r_0: \quad E_r(r) = 0 = E_\theta(r) \quad . \tag{112c}$$

The dispersion relations are found by applying the following boundary conditions: E_θ and $E'_\theta + (i\ell/r_0)E_r$ are continuous at r_0. {It is easily shown, using Eqs. (109a-c), that the additional boundary condition found in Section IV that remains when $E_z = 0$, namely $\Delta[\omega^2 - \sigma_1^2 - (kV_0/\Omega)B_1]E_r = 0$ is no longer independent of $\Delta E_\theta = 0$ and $\Delta[E'_\theta + (i\ell/r)E_r] = 0$}.

The dispersion relations are

$$P_{5S} = 0 \tag{113a}$$

$$\eta_S J_\ell(\eta_S r_0) = 0 \tag{113b}$$

$$\eta_S^2 J'_\ell(\eta_S r_0) + \frac{\ell^2}{r_0^2} J_\ell(\eta_S r_0) = 0 \quad . \tag{113c}$$

However Eqs. (113b,c) are satisfied only if $\eta_S = 0$ which implies that the fields are zero everywhere. It is easily shown that this result is also obtained when $P_{5L} = 0$ and for the cases for which the plasma is finite.

Hence it can be concluded that pure transverse waves are not possible for $\ell \neq 0$.

VIII CONCLUDING REMARKS

The main results of this paper are the dispersion relations derived in Sections V, VI and VII. That is, Eqs. (87a-f), (89a-e), (91a-g), (94) and (102). In the next report of this series, these dispersion relations will be analyzed for possible unstable solutions.

Further extension of the analysis presented here of the beam-plasma system is contemplated in which, for example, external electromagnetic fields and non-uniform densities of the beam and/or plasma are considered.

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APPENDIX

VECTOR IDENTITIES IN CYLINDRICAL COORDINATES

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VECTOR IDENTITIES IN CYLINDRICAL COORDINATES

Listed in this appendix are the vector identities necessary in order to obtain the beam and plasma current responses, Eqs. (57) and (63). These identities are given in cylindrical coordinates using the notation $E' = dE/dr$, $E'' = d^2E/dr^2$, etc. The space-time dependence of the electric field is assumed to be

$$\vec{E}(x,y,z,t) = \vec{E}(r)e^{i(kz-\omega t+\ell\theta)} \tag{A.1}$$

where

$$r = [x^2 + y^2]^{1/2} .$$

Let

$$\vec{F}(r,\vec{v}) = \vec{E}(r) + \frac{\vec{V}_0 + \vec{v}}{i\omega} \times [(\nabla_\perp + ik\hat{z}) \times \vec{E}(r)] \tag{A.2}$$

where the exponential factor, $e^{i(kz-\omega t+\ell\theta)}$, is understood on both sides of (A.1) and in all the formulas that follow. For the plasma set $V_0 = 0$. Thus

$$\begin{aligned}\vec{F}(r,\vec{v}) = {} & \hat{r}\left[\frac{e - kv_z}{\omega}E_r + \frac{V_0 + v_z}{i\omega}E_z' + \frac{v_\theta}{i\omega}\left(E_\theta' + \frac{1}{r}E_\theta - \frac{i\ell}{r}E_r\right)\right] \\ & + \hat{\theta}\left[\frac{\Omega - kv_z}{\omega}E_\theta + \frac{V_0 + v_z}{i\omega}\frac{i\ell}{r}E_z - \frac{v_r}{i\omega}\left(E_\theta' + \frac{1}{r}E_\theta - \frac{i\ell}{r}E_r\right)\right] \\ & + \hat{z}\left[E_z + \frac{v_r}{i\omega}(ikE_r - E_z') + \frac{v_\theta}{i\omega}\left(ikE_\theta - \frac{i\ell}{r}E_z\right)\right]\end{aligned} \tag{A.3}$$

$$(\vec{r}_0 \cdot \nabla_v)\vec{F} = \hat{\theta}\left[-\frac{r}{i\omega}\left(E_\theta' + \frac{1}{r}E_\theta - \frac{i\ell}{r}E_r\right)\right] + \hat{z}\left[\frac{r}{i\omega}(ikE_r - E_z')\right] . \tag{A.4}$$

$$(\vec{v}_\perp \cdot \nabla_v)\vec{F} = \hat{r}\left[\frac{v_\theta}{i\omega}\left(E'_\theta + \frac{1}{r}E_\theta - \frac{i\ell}{r}E_r\right)\right]$$

$$+ \hat{\theta}\left[-\frac{v_r}{i\omega}\left(E'_\theta + \frac{1}{r}E_\theta - \frac{i\ell}{r}E_r\right)\right]$$

$$+ \hat{z}\left[\frac{v_r}{i\omega}(ikE_r - E'_z) + \frac{v_\theta}{i\omega}\left(ikE_\theta - \frac{i\ell}{r}E_z\right)\right]. \qquad \text{(A.5)}$$

$$(\vec{v}_\perp \cdot \nabla)\vec{F} = \hat{r}\Bigg[v_r E'_r + \frac{v_\theta}{r}(i\ell E_r - E_\theta) + \frac{v_r v_\theta}{i\omega}\left(E''_\theta + \frac{2}{r}E'_\theta - \frac{i\ell}{r}E'_r\right)$$

$$+ \frac{v_r(V_0+v_z)}{i\omega}(E''_z - ikE'_r) + \frac{v_\theta(V_0+v_z)}{i\omega r}\left\{ikE_\theta + i\ell\left(E'_z - \frac{1}{r}E_z - ikE_r\right)\right\}$$

$$+ \frac{v_\theta^2}{i\omega r}\, i\ell\left(E'_\theta + \frac{1}{r}E_\theta - \frac{i\ell}{r}E_r\right)\Bigg]$$

$$+ \hat{\theta}\Bigg[v_r E'_\theta + \frac{v_\theta}{r}(i\ell E_\theta + E_r) + \frac{v_r^2}{i\omega}\left\{-E''_\theta - \frac{1}{r}E'_\theta + \frac{1}{r^2}E_\theta + \frac{i\ell}{r}\left(E'_r - \frac{1}{r}E_r\right)\right\}$$

$$+ \frac{v_r v_\theta}{i\omega r}\, i\ell\left(-E'_\theta - \frac{1}{r}E_\theta + \frac{i\ell}{r}E_r\right) + \frac{v_r(V_0+v_z)}{i\omega}\left\{\frac{i\ell}{r}\left(E'_z - \frac{1}{r}E_z\right) - ikE'_\theta\right\}$$

$$+ \frac{v_\theta(V_0+v_z)}{i\omega r}\left\{i\ell\left(\frac{i\ell}{r}E_z - ikE_\theta\right) + E'_z - ikE_r\right\} + \frac{v_\theta^2}{i\omega r}\left(E'_\theta + \frac{1}{r}E_\theta - \frac{i\ell}{r}E_r\right)\Bigg]$$

$$+ \hat{z}\Bigg[v_r E'_z + v_\theta\frac{i\ell}{r}E_z + \frac{v_r^2}{i\omega}\left(ikE'_r - E''_z\right) + \frac{v_r v_\theta}{i\omega}\Bigg\{-\frac{i\ell}{r}\left(2E'_z - \frac{1}{r}E_z\right)$$

$$- \frac{\ell k}{r}E_r + ikE'_\theta\Bigg\} + \frac{v_\theta^2}{\omega r}\,\ell\left(-\frac{i\ell}{r}E_z + ikE_\theta\right)\Bigg] \qquad \text{(A.6)}$$

$$(\vec{r}\cdot\nabla)\vec{F} = \hat{r}r\left[\frac{\Omega - kv_z}{\omega}E_r' + \frac{V_0+v_z}{i\omega}E_z'' + \frac{v_\theta}{i\omega}\left\{E_\theta'' + \frac{1}{r}E_\theta' - \frac{1}{r^2}E_\theta - \frac{i\ell}{r}\left(E_r' - \frac{1}{r}E_r\right)\right\}\right]$$

$$+ \hat{\theta}r\left[\frac{\Omega - kv_z}{\omega}E_\theta' + \frac{V_0+v_z}{i\omega}\frac{i\ell}{r}\left(E_z' - \frac{1}{r}E_z\right) - \frac{v_r}{i\omega}\left\{E_\theta'' + \frac{1}{r}E_\theta'\right.\right.$$

$$\left.\left.- \frac{1}{r^2}E_\theta - \frac{i\ell}{r}\left(E_r' - \frac{1}{r}E_r\right)\right\}\right]$$

$$+ \hat{z}r\left[E_z' + \frac{v_r}{i\omega}\left(ikE_r' - E_z''\right) + \frac{v_\theta}{i\omega}\left\{ikE_\theta' - \frac{i\ell}{r}\left(E_z' - \frac{1}{r}E_z\right)\right\}\right] \qquad (A.7)$$

The following is only needed for the plasma, so V_0 has been set equal to zero:

$$(\vec{v}_\perp\cdot\nabla)^2\vec{F} = \hat{r}\left[v_r^2E_z'' - \frac{v_\theta^2}{r^2}\{(1+\ell^2)E_r + 2i\ell E_\theta\} + v_rv_\theta\left\{\frac{i\ell}{r}\left(2E_r' - \frac{1}{r}E_r\right)\right.\right.$$

$$\left.- \frac{1}{r}\left(2E_\theta' - \frac{1}{r}E_\theta\right)\right\} + v_r^2v_\theta\left\{E_\theta''' + \frac{3}{r}E_\theta'' - \frac{1}{r^2}E_\theta' - \frac{1}{r^3}E_\theta - \frac{i\ell}{r}\left(E_r'' - \frac{1}{r}E_r\right)\right\}$$

$$+ v_r^2v_z(E_z''' - ikE_r'') + v_rv_\theta v_z\frac{1}{r}\left\{ik\left(2E_\theta' - \frac{1}{r}E_\theta\right) + \ell k\left(2E_r' - \frac{1}{r}E_r\right)\right.$$

$$\left.+ i\ell\left(2E_z'' - \frac{3}{r}E_z' + \frac{3}{r^2}E_z\right)\right\} + v_rv_\theta^2\left\{\frac{\ell^2}{r^2}\left(2E_r' - \frac{1}{r}E_r\right)\right.$$

$$\left.+ \frac{i\ell}{r}\left(2E_\theta'' + \frac{3}{r}E_\theta' - \frac{1}{r^2}E_\theta\right)\right\} + v_zv_\theta^2\left\{-\frac{\ell^2+1}{r^2}E_z' + \frac{2\ell^2}{r^3}E_z\right.$$

$$\left.\left.+ \frac{ik}{r^2}(1+\ell^2)E_r - \frac{\ell k}{r^2}E_\theta\right\} + v_\theta^3\left\{-\frac{\ell^2+1}{r^2}\left(E_\theta' + \frac{1}{r}E_\theta - \frac{i\ell}{r}E_r\right)\right\}\right] \qquad (A.8)$$

$$+\hat{\theta}\left[v_r^2E_\theta'' - \frac{v_\theta^2}{r^2}\left\{(1+\ell^2)E_\theta - 2i\ell E_r\right\} + v_r v_\theta \left\{\frac{i\ell}{r}\left(2E_\theta' - \frac{1}{r}E_\theta\right)\right.\right.$$

$$\left.+\frac{1}{r}\left(2E_r' - \frac{1}{r}E_r\right)\right\} - v_r^3\left\{E_\theta''' + \frac{1}{r}E_\theta'' - \frac{2}{r^2}E_\theta' + \frac{2}{r^3}E_\theta - \frac{i\ell}{r}\left(E_r'' - \frac{2}{r}E_r'\right.\right.$$

$$\left.\left.+\frac{2}{r^2}E_r\right)\right\} + v_r^2 v_\theta\left\{-\frac{i\ell}{r}\left(2E_\theta'' + \frac{1}{r}E_\theta' - \frac{3}{r^2}E_\theta\right) - \frac{\ell^2}{r^2}\left(2E_r' - \frac{3}{r}E_r\right)\right\}$$

$$+v_r^2 v_z\left\{\frac{i\ell}{r}\left(E_z'' - \frac{2}{r}E_z' + \frac{2}{r^2}E_z\right) - ikE_\theta''\right\} + v_r v_\theta v_z\left\{E_z''' + \frac{1}{r}E_z'' - 2\frac{\ell^2}{r^2}E_z'\right.$$

$$\left.+3\frac{\ell^2}{r^3}E_z - ik\left(E_r'' + \frac{1}{r}E_r'\right) + \frac{\ell k}{r}\left(2E_\theta' - \frac{1}{r}E_\theta\right)\right\} + v_r v_\theta^2\left\{\frac{2}{r}E_\theta'' + \frac{\ell^2+2}{r^2}E_\theta'\right.$$

$$\left.+\frac{\ell^2-2}{r^3}E_\theta - \frac{i\ell}{r^2}\left(2E_r' + \frac{\ell^2-2}{r}E_r\right)\right\} + v_\theta^2 v_z\left\{\frac{i\ell}{r^2}\left(2E_z' - \frac{\ell^2+1}{r^2}E_z\right)\right.$$

$$\left.\left.+2\frac{\ell k}{r^2}E_r + ik\frac{\ell^2+1}{r^2}E_\theta\right\} + v_\theta^3\frac{2i\ell}{r^2}\left(E_\theta' + \frac{1}{r}E_\theta - \frac{i\ell}{r}E_r\right)\right]$$

$$\hat{z}\left[v_r^2E_z'' - v_\theta^2\frac{\ell^2}{r^2}E_z + v_r v_\theta\frac{i\ell}{r}\left(2E_z' - \frac{1}{r}E_z\right)\right.$$

$$+v_3^3(ikE_r'' - E_z''') + v_r^2 v_\theta\left\{ikE_\theta'' - \frac{i\ell}{r}\left(3E_z'' - \frac{3}{r}E_z' + \frac{2}{r^2}E_z\right)\right.$$

$$\left.-\frac{\ell k}{r}\left(2E_r' - \frac{1}{r}E_r\right)\right\} + v_r v_\theta^2\left\{\frac{3\ell^2}{r^2}\left(E_z' - \frac{1}{r}E_z\right) - ik\frac{\ell^2}{r^2}E_r\right.$$

$$\left.\left.-\frac{\ell k}{r}\left(2E_\theta' - \frac{1}{r}E_\theta\right)\right\} + v_\theta^3\left\{\frac{i\ell^3}{r^3}E_z - \frac{ik\ell^2}{r^2}E_\theta\right\}\right] . \qquad \text{(A.8) (cont)}$$